Explicit Formulas for Local Formal Mellin Transforms

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Abstract

ADAM GRAHAM-SQUIRE: Explicit Formulas for Local Formal Mellin Transforms

(Under the direction of Dima Arinkin)

Much recent work has been done on the local Fourier transforms for connections on the punctured formal disk. Specifically, the local Fourier transforms have been introduced, shown to induce certain equivalences of categories, and explicit formulas have been found to calculate them. Our goal is to corroborate recent results for calculation of the local Fourier transforms and then extend our methods to a similar situation, the local Mellin transforms.

This dissertation is divided into three main parts. In the first part we prove explicit formulas for calculation of the local Fourier transforms. These formulas have recently been proved by others, and we reproduce their results using different techniques. The other two parts of the dissertation are given over to applying those same techniques to the local Mellin transforms for connections on the punctured formal disk. In the second part, we introduce the local Mellin transforms and show that they induce equivalences between certain categories of vector spaces with connection and vector spaces with invertible difference operators. In the third part we find formulas for explicit calculation of the local Mellin transforms in the same spirit as the results for the local Fourier transforms.

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Introduction

The main goal of this dissertation is the definition and calculation of the local Mellin transforms on a punctured formal disk. In order to understand the local Mellin transforms, it is necessary to understand recent work that has been done on an analogous construction, that of the local Fourier transforms for connections on the formal punctured disk. In this introduction, we wish to put our work on the local Mellin transforms in its proper context by explaining the genesis and the recent history of research done on the local Fourier transforms. In section 0.1 we give a brief historical background to the local Fourier transforms. We describe in section 0.2 recent work that has been done on the local Fourier transforms, and in section 0.3 we explain the theoretical origin of the local Mellin transforms and what we wish to prove about them. Section 0.4 provides a thumbnail sketch of the main body of the dissertation.

0.1. Historical context

0.1.1. Classical Fourier transform. In order to understand the origin of the *local* Fourier transform, our narrative begins with the 'classical' Fourier transform. For a suitable function f, the Fourier transform of f is given by

$$\hat{f}(\xi) = \int f(x)e^{-ix\xi}dx.$$

In particular, the Fourier transform has two main properties: it is invertible (and the inverse is of virtually the same form) and the following equalities hold

- $\frac{d}{d\xi}\hat{f} = \widehat{ixf}$
- $\xi\hat{f} = -i\widehat{\frac{d}{dx}f}$.

The equalities above can be useful in the following manner: one can apply the Fourier transform to an ordinary differential equation and then solve the resulting, hopefully easier, differential equation. In a related form, one can also consider the Fourier transform as an action on a differential operator, which is described below.

0.1.2. Fourier transform on the Weyl algebra. Consider the ring of ordinary differential operators with polynomial coefficients, also known as the Weyl algebra. We write $\mathbb{W}$ for the Weyl algebra, and thus we have

$$\mathbb{W} = \mathbb{C}[z]\left\langle \frac{d}{dz} \right\rangle = \left\{ p_n(z)\frac{d}{dz}^n + \cdots + p_0(z) \middle| p_i(z) \in \mathbb{C}[z] \right\}$$

with the convention that $\frac{d}{dz}z = z\frac{d}{dz} + 1$. In this context, the Fourier transform is the automorphism $FT : \mathbb{W} \to \mathbb{W}$ defined on its generators by $z \mapsto \frac{d}{dz}$ and $\frac{d}{dz} \mapsto -z$. What we refer to as the *local* Fourier transforms can be derived from the Fourier transform on the Weyl algebra as we explain in the following subsection. We note that the description below is only for motivation; it is a framework and is not written in a completely rigorous manner. Definitions of the local Fourier transforms are given a precise construction in the body of the dissertation.

0.1.3. Local Fourier transform. For every element p of the Weyl algebra, one can "divide through by the leading coefficient" to get a monic element of the algebra $\mathbb{C}(z)\left\langle \frac{d}{dz} \right\rangle$, that is, a polynomial in $\frac{d}{dz}$ with coefficients that are rational functions in z. This allows one to associate to every element of $\mathbb{W}$ a unique element of $\mathbb{C}(z)\left\langle \frac{d}{dz} \right\rangle$. By writing a higher order differential operator as a system of linear differential operators, one can associate to each element of $\mathbb{C}(z)\left\langle \frac{d}{dz} \right\rangle$ a linear *matrix* differential operator with entries in $\mathbb{C}(z)$ (for details of this process, see [**vdPS03**, Section 1.2]). If one is interested in looking *locally* at a linear matrix differential operator with entries in $\mathbb{C}(z)$, one can choose a local coordinate and consider the corresponding linear matrix differential operator with entries that are formal Laurent series, which we denote as $\mathbb{C}((z))$. Thus to each $p \in \mathbb{W}$

and $x \in \mathbb{C}$ one can associate P_x, a linear matrix differential operator with entries in $\mathbb{C}((z))$.

In a similar fashion, to the original differential operator $p \in \mathbb{W}$, one can apply the Fourier transform to get $FT(p) = \hat{p} \in \mathbb{W}$. One can then apply the process described above to $\hat{p}$ in order to get a corresponding linear matrix differential operator with entries in $\mathbb{C}((z))$, call it $\hat{P}_y$. The *local* Fourier transform (LFT) is the map that takes the equivalence class of P_x to the equivalence class of $\hat{P}_y$ and completes the diagram given below:

$$\begin{tikzcd}
p_n(z)\frac{d}{dz}^n + \cdots + p_0(z) \arrow[r, "FT"] \arrow[d] & \hat{p}_m(z)\frac{d}{dz}^m + \cdots + \hat{p}_0(z) \arrow[d] \\
\frac{d}{dz}^n + \cdots + q_0(z) \arrow[d] & \frac{d}{dz}^m + \cdots + \hat{q}_0(z) \arrow[d] \\
\frac{d}{dz} + A \arrow[d] & \frac{d}{dz} + \hat{A} \arrow[d] \\
\frac{d}{dz} + L/\sim \arrow[r, dotted, "LFT"] & \frac{d}{dz} + \hat{L}/\sim
\end{tikzcd}$$

Here p corresponds to $p_n(z)\frac{d}{dz}^n + \cdots + p_0(z)$ with $p_i \in \mathbb{C}[z]$ (respectively $\hat{p}_i$) and $q_i \in \mathbb{C}(z)$ (respectively $\hat{q}_i$). We also have $A \in M_n\left(\mathbb{C}(z)\left\langle \frac{d}{dz} \right\rangle\right)$, $\hat{A} \in M_m\left(\mathbb{C}(z)\left\langle \frac{d}{dz} \right\rangle\right)$, $L \in M_n\left(\mathbb{C}((z))\left\langle \frac{d}{dz} \right\rangle\right)$, $\hat{L} \in M_m\left(\mathbb{C}((z))\left\langle \frac{d}{dz} \right\rangle\right)$, and the equivalence for the matrix differential operators refers to *gauge* equivalence. It should not be clear at this point that the map LFT even exists or is well-defined. We describe in the next section the recent work that has been done to put the local Fourier transform on a rigorous footing.

Remark. The local Fourier transform has different flavors depending on the point of localization, hence in this dissertation we generally refer to local Fourier transform*s*. Both conventions for terminology are found in the literature. In general, the study of local Fourier transforms need not be restricted to the complex numbers. Thus in later chapters we work over an arbitrary algebraically closed field of characteristic zero.

0.2. Local Fourier transform

0.2.1. Local Fourier for ℓ-adic sheaves. The seminal paper regarding the local Fourier transform is [**Lau87**] by G. Laumon in 1987. To study the local behavior of the Fourier transform, Laumon introduced the geometric stationary phase principle and established the local Fourier transformations. The transformations as given in [**Lau87**] are defined in terms of cohomological functors and thus are rarely computable, but in [**Lau87**, 2.6.3], Laumon and Malgrange give conjectural formulas of local Fourier transforms for a class of $\bar{\mathbb{Q}}_\ell$-sheaves, which were later proved (with a slight adjustment) by L. Fu in [**Fu07**].

0.2.2. Local Fourier transforms for meromorphic connections over Laurent series fields. In the case of formal differential systems defined over a field of characteristic zero, the analogues of the local Fourier transforms defined by Laumon were given independently in 2004 by S. Bloch and H. Esnault in [**BE04**] and by R. Garcia Lopez in [**GL04**]. The methods used in [**BE04**] and [**GL04**] are different, though. The construction of the local Fourier transforms in [**GL04**] follows the microlocal techniques of B. Malgrange. In [**Mal81**], Malgrange gives a microanalytic construction for the local Fourier transforms $\mathcal{F}^{(0,\infty)}$ and $\mathcal{F}^{(\infty,0)}$, which Garcia Lopez then extends in [**GL04**]. In [**BE04**], rigorous definitions for the local Fourier transforms for meromorphic connections over Laurent series fields are given and it is shown that they have properties analogous to the local ℓ-adic Fourier transforms. However, neither [**BE04**] nor [**GL04**] give explicit formulas for how to calculate the local Fourier transforms.

0.2.3. Explicit formulas for local Fourier transforms. Calculation of the local Fourier transforms was proved independently in 2007 by C. Sabbah in [**Sab07**] and J. Fang in [**Fan07**]. The calculations were of a quite different flavor, with Sabbah taking a geometric approach and Fang's proof more algebraic in nature. An important technical tool used in the calculation is the formal reduction of differential operators to a canonical form. This work goes back to H.L. Turritin [**Tur55**] and A. Levelt [**Lev75**], with more recent expositions given by B. Malgrange [**Mal91**], D.G. Babbitt and V.S. Varadarajan [**BV85**], as well as M. van der Put and M. Singer [**vdPS03**]. In [**BBE**, Section 5.9], Beilinson, Bloch and Esnault present the canonical form in a way that is useful to our construction, and in [**Var96**] Varadarajan gives a helpful historical overview and summary of research on linear meromorphic differential equations.

0.2.4. Another viewpoint. In 2008 D. Arinkin's paper [**Ari**] gave another framework for the local Fourier transforms. His construction is beneficial because it makes relating the singularities of a holonomic $\mathcal{D}$-module and its global Fourier transform virtually tautological. Such a relation is one of the central properties of the local Fourier transforms. Arinkin goes on to give explicit calculation of the Katz-Radon transform in [**Ari**] as well. In Chapter 2, we use the same methods of [**Ari**] to give another proof of the calculations done in [**Fan07**] and [**Sab07**]. Arinkin's techniques are particularly advantageous because they generalize nicely to making calculations for local Mellin transforms.

0.3. Local Mellin transform

The general motivation for the local Mellin transform is virtually identical to the motivation given above for the local Fourier transform, so we do not repeat it in full detail here. We mention only that the 'classical' Mellin transform can be stated as follows: for an appropriate f the Mellin transform of f is given by

$$\tilde{f}(\eta) = \int_0^\infty x^{\eta-1} f(x)dx$$

and one can check that the following identities hold:

- $\eta\tilde{f} = -\widetilde{\left(x\frac{d}{dx}f\right)}$
- $\Phi\tilde{f} = \widetilde{(xf)}$

where Φ is the difference operator taking $\tilde{f}(\eta)$ to $\tilde{f}(\eta+1)$. These identities can be used to define the local Mellin transform in the same way that properties of the Fourier transform are used to define the local Fourier transform. As in the case of the Fourier transform, the local Mellin transform has different 'flavors' depending on the point of singularity, so we refer to them as local Mellin transform*s*.

There is one large distinction between the local Fourier transforms and the local Mellin transforms, however. Whereas the local Fourier transforms take a linear matrix differential operator to another linear matrix *differential* operator, the local Mellin transforms take a linear matrix differential operator to a linear matrix *difference* operator. In [**BE04**] and [**GL04**], clear connections are drawn between the local and global Fourier transforms. Such connections can be made for the Mellin transform as well, but are not included in this dissertation as our focus is purely on the local construction.

The *global* Mellin transform for connections on a punctured formal disk is given by Laumon in [**Lau96**], but since that time little work has been done on the Mellin transform in this area. In [**Ari**, Section 2.5], Arinkin remarks that it would be interesting to apply his methods to other integral transforms such as the Mellin transform. Chapters 3 and 4 of this dissertation are the answer to that query. In Chapter 3, we define the local Mellin transforms in an analogous manner to the definitions of the local Fourier transforms which were given in [**Ari**], [**BE04**], and [**GL04**]. In particular, we mimic the framework given in [**Ari**] in order to define the local Mellin transforms, as Arinkin's construction lends itself most easily to calculation. In Chapter 3 we also show that the local Mellin transforms induce equivalences for certain categories of vector spaces with connection and vector spaces with difference operators. Such equivalences could, in principle, reduce questions about difference operators to questions about (relatively more-studied) connections, although we do not do such an analysis in this work. In Chapter 4 we give explicit formulas for the local Mellin transforms in the spirit of those given in Chapter 2 (as well as [**Fan07**] and [**Sab07**]) for the local Fourier transform. Again an important tool for the calculation will be the formal reduction of differential operators described above, as well as the formal reduction of linear difference operators given by C. Praagman in [**Pra83**] as well as G. Chen and A. Fahim in [**CF98**]. There are considerable parallels between difference operators and connections, and we refer the reader to [**vdPS97**] for more details.

0.4. Overview of dissertation

A brief description of the remainder of the dissertation is as follows: In Chapter 1 we give a synopsis of the most important notation, conventions, results and definitions that we will use. Chapter 2 is devoted to proving explicit formulas for calculating the local Fourier transforms for connections on a formal punctured disk (e. g. Theorem 2.2.1), reproducing the results of [**Fan07**] and [**Sab07**] with a different method of proof. In Chapter 3 we introduce the local Mellin transforms (e. g. Definition 3.3.1) and their inverses and describe some of their properties, then we give explicit formulas for the local Mellin transforms in Chapter 4 (e. g. Theorem 4.1.1).

CHAPTER 1

Background

In this chapter we give the definitions, notation, and results most pertinent to the dissertation. We also describe recent research related to our topic so as to put our work in its proper context in a rigorous manner. In section 1.1 we discuss connections on the punctured formal disk and in section 1.2 we give an analogous construction for difference operators on the punctured formal disk. Section 1.3 details some useful notation we will use throughout the dissertation and section 1.4 has information about the norm and order of an operator. We end with section 1.5, where we prove a lemma that will be important for our calculations in Chapters 2 and 4.

We fix a ground field $\Bbbk$, which is assumed to be algebraically closed of characteristic zero.

1.1. Connections on the formal disk

One of the primary objects we work with are differential operators, or connections, on the formal disk. Specifically, the local Fourier transform takes a connection on the formal disk and outputs a connection on the formal disk. The local Mellin transform inputs a connection and produces a difference operator on the formal disk, which we define in section 1.2. We write $K = \Bbbk((z))$ for the field of formal Laurent series.

1.1.1. Definitions.

Definition 1.1.1. Let V be a finite-dimensional vector space over K. A *connection* on V is a $\Bbbk$-linear operator $\nabla : V \to V$ satisfying the Leibniz identity:

$$\nabla(fv) = f\nabla(v) + \frac{df}{dz}v$$

for all $f \in K$ and $v \in V$. Equivalently, we can write that $[\nabla, f] := \nabla f - f\nabla = f'$. A choice of basis in V gives an isomorphism $V \simeq K^n$; we can then write ∇ as $\frac{d}{dz} + A$, where $A = A(z) \in \mathfrak{gl}_n(K)$ is the *matrix* of ∇ with respect to this basis. We sometimes refer to the $\frac{d}{dz}$ (respectively A) as the *differential* part (respectively *linear* part) of the operator ∇.

Definition 1.1.2. We write $\mathcal{C}$ for the *category of vector spaces with connections over* K. Its objects are pairs (V, ∇), where V is a finite-dimensional K-vector space and $\nabla : V \to V$ is a connection. Morphisms between (V_1, ∇_1) and (V_2, ∇_2) are K-linear maps $\phi : V_1 \to V_2$ that are *horizontal* in the sense that $\phi\nabla_1 = \nabla_2\phi$.

1.1.2. Properties of connections. We summarize below some well-known properties of connections on the formal disk. The results go back to Turritin [**Tur55**] and Levelt [**Lev75**]; more recent references include [**BV85**], [**BBE**, Sections 5.9 and 5.10], [**Mal91**], and [**vdPS97**].

Let q be a positive integer and consider the field $K_q = \Bbbk((z^{1/q}))$. Note that K_q is the unique extension of K of degree q. For every $f \in K_q$, we define an object $E_f \in \mathcal{C}$ by

$$E_f = E_{f,q} = \left(K_q, \frac{d}{dz} + z^{-1}f\right).$$

In terms of the isomorphism class of an object E_f, the reduction procedures of [**Tur55**] and [**Lev75**] imply that we need only consider f in the quotient

$$\Bbbk((z^{1/q}))\Big/\left(z^{1/q}\Bbbk[[z^{1/q}]] + \frac{1}{q}\mathbb{Z}\right) \tag{1.1}$$

where $\Bbbk[[z]]$ denotes formal *power* series.

Let R_q (we write $R_q(z)$ when we wish to emphasize the local coordinate) be the set of orbits for the action of the Galois group $\mathrm{Gal}(K_q/K)$ on the quotient. Explicitly, the Galois group is identified with the group of degree q roots of unity $\eta \in \Bbbk$; the action on $f \in R_q$ is by $f(z^{1/q}) \mapsto f(\eta z^{1/q})$. Finally, denote by $R_q^\circ \subset R_q$ the set of $f \in R_q$ that cannot be represented by elements of K_r for any $0 < r < q$.

Remark. R_q° can alternatively be described as the locus of R_q where $\mathrm{Gal}(K_q/K)$ acts freely.

Definition 1.1.3. We define the direct sum $\oplus$ and tensor product $\otimes$ on the category $\mathcal{C}$ as follows:

$$(V_1, \nabla_1) \oplus (V_2, \nabla_2) := (V_1 \oplus V_2, \nabla_+),$$

where $\nabla_+(v_1 \oplus v_2) = \nabla_1(v_1) \oplus \nabla(v_2)$.

$$(V_1, \nabla_1) \otimes (V_2, \nabla_2) := (V_1 \otimes V_2, \nabla_\times),$$

where $\nabla_\times(v_1 \otimes v_2) = \nabla_1(v_1) \otimes v_2 + v_1 \otimes \nabla_2(v_2)$.

Proposition 1.1.4.

(1) The isomorphism class of E_f depends only on the orbit of the image of f in R_q.

(2) E_f is irreducible if and only if the image of f in R_q belongs to R_q°. As q and f vary, we obtain a complete list of isomorphism classes of irreducible objects of $\mathcal{C}$.

(3) Every $E \in \mathcal{C}$ can be written as

$$E \simeq \bigoplus_i (E_{f_i,q_i} \otimes J_{m_i}),$$

where the $E_{f,q}$ are irreducible, $J_m = (K^m, \frac{d}{dz} + z^{-1}N_m)$, and N_m is the nilpotent Jordan block of size m.

Proofs of the proposition are either prevalent in the literature (cf. [**BBE**], [**Mal91**], [**vdPS97**]) or straightforward and thus are omitted here. Analogous properties of difference operators are given in the following section, and we give proofs at that point.

Remark. We often refer to the objects $(E_f \otimes J_m) \in \mathcal{C}$ as *indecomposable* objects in $\mathcal{C}$.

1.2. Difference operators on the formal disk

Vector spaces with difference operator and vector spaces with connection are defined in a similar fashion.

1.2.1. Definitions.

Definition 1.2.1. Let V be a finite-dimensional vector space over $K = \Bbbk((\theta))$. A *difference operator* on V is a $\Bbbk$-linear operator $\Phi : V \to V$ satisfying

$$\Phi(fv) = \varphi(f)\Phi(v)$$

for all $f \in K$, $v \in V$, with $\varphi : K^n \to K^n$ as the $\Bbbk$-automorphism defined below. A choice of basis in V gives an isomorphism $V \simeq K^n$; we can then write Φ as $A\varphi$, where $A = A(\theta) \in \mathfrak{gl}_n(K)$ is the *matrix* of Φ with respect to this basis, and for $v(\theta) \in K^n$ we have

$$\varphi(v(\theta)) = v\left(\frac{\theta}{1+\theta}\right) = v\left(\sum_{i=1}^{\infty}(-1)^{i+1}\theta^i\right).$$

We follow the convention of [**Pra83**, Section 1] to define φ over the extension $K_q = \Bbbk((\theta^{1/q}))$. Thus for all $q \in \mathbb{Z}^+$, φ extends to a $\Bbbk$-automorphism of K_q^n defined by

$$\varphi\Big(v(\theta^{1/q})\Big) = v\left(\theta^{1/q}\left[\sum_{i=0}^{\infty}\binom{-1/q}{i}\theta^i\right]\right).$$

Definition 1.2.2. We write $\mathcal{N}$ for the *category of vector spaces with invertible difference operator over* K. Its objects are pairs (V, Φ), where V is a finite-dimensional K-vector space and $\Phi : V \to V$ is an invertible difference operator. Morphisms between (V_1, Φ_1) and (V_2, Φ_2) are K-linear maps $\phi : V_1 \to V_2$ such that $\phi\Phi_1 = \Phi_2\phi$.

1.2.2. Properties of difference operators. In [**CF98**] and [**Pra83**], a canonical form for difference operators is constructed. We give an equivalent construction in the theorem below, which is a restatement of [**Pra83**, Theorem 8 and Corollary 9] with different notation so as to better fit our situation.

Theorem 1.2.3 ([**Pra83**], Theorem 8 and Corollary 9)**.** *Let $\Phi : V \to V$ be an invertible difference operator.*

Then there exists a finite (Galois) extension L of K and a basis of $L \otimes_K V$ such that Φ is expressed as a diagonal block matrix. Each block is of the form

$$F_g = \begin{bmatrix} g & & \\ \theta^{\lambda+1} & \ddots & \\ & \ddots & \ddots \end{bmatrix}$$

with $g \in K_q$, $\lambda \in \frac{1}{q}\mathbb{Z}$, $g = a_0\theta^\lambda + \cdots + a_q\theta^{\lambda+1}$, $a_0 \neq 0$, and a_q defined up to a shift by $\frac{a_0}{q}\mathbb{Z}\theta^{\lambda+1}$. The matrix is unique modulo the order of the blocks.

Remark. The F_g are the indecomposable components for the matrix of Φ.

Theorem 1.2.3 allows us to describe the category $\mathcal{N}$ in a fashion similar to our description of the category $\mathcal{C}$. For every $g \in K_q$, we define an object $D_g \in \mathcal{N}$ by

$$D_g = D_{g,q} := \Big(K_q, g\varphi\Big).$$

The canonical form given in Theorem 1.2.3 implies that we need only consider g in the following quotient of the multiplicative group $\Bbbk((\theta^{1/q}))^*$:

$$K_q^* \Big/ \left(1 + \frac{1}{q}\mathbb{Z}\theta + \theta^{1+(1/q)}\Bbbk[[\theta^{1/q}]]\right). \tag{1.2}$$

Let S_q be the set of orbits for the action of the Galois group $\mathrm{Gal}(K_q/K)$ on the quotient given in (1.2). Denote by $S_q^\circ \subset S_q$ the set of $g \in S_q$ that cannot be represented by elements of K_r for any $0 < r < q$. As before, S_q° can be thought of as the locus where $\mathrm{Gal}(K_q/K)$ acts freely.

Definition 1.2.4. We define the direct sum $\oplus$ and tensor product $\otimes$ on the category $\mathcal{N}$ as follows:

$$(V_1, \Phi_1) \oplus (V_2, \Phi_2) := (V_1 \oplus V_2, \Phi_+),$$

where $\Phi_+(v_1 \oplus v_2) = \Phi_1(v_1) \oplus \Phi(v_2)$.

$$(V_1, \Phi_1) \otimes (V_2, \Phi_2) := (V_1 \otimes V_2, \Phi_\times),$$

where $\Phi_\times(v_1 \otimes v_2) = \Phi_1(v_1) \otimes \Phi_2(v_2)$.

Proposition 1.2.5.

(1) The isomorphism class of D_g depends only on the orbit of the image of g in S_q.

(2) D_g is irreducible if and only if the image of g in S_q belongs to S_q°. As q and g vary, we obtain a complete list of isomorphism classes of irreducible objects of $\mathcal{N}$.

(3) Every $D \in \mathcal{N}$ can be written as

$$D \simeq \bigoplus_i (D_{g_i,q_i} \otimes T_{m_i}),$$

where the $D_{g,q}$ are irreducible, $T_m = (K^m, U_m\varphi)$, and $U_m = I_m + \theta N_m$ where N_m is the nilpotent Jordan block of size m.

Proof. (1) For a given object $D_{g'}$, Theorem 1.2.3 implies that $D_{g'} \simeq D_g$ where $g' \equiv g$ as elements of the quotient given in (1.2). All that is left is to show that an action of the Galois group does not affect the isomorphism class. Let $\mu \in \Bbbk$ such that $\mu^q = 1$. Consider the morphism $\phi : K_q \to K_q$ defined by $\phi(\theta^{1/q}) = \mu\theta^{1/q}$. Then ϕ is a K-linear map that is easily seen to be an isomorphism sending $D_{g(\theta^{1/q})}$ to $D_{g(\mu\theta^{1/q})}$.

(2) Note that an object $D = (V, \Phi)$ is irreducible if the only Φ-invariant subspaces of V are 0 and V. A proof of the first statement is as follows. To prove the forward direction, let $D_g \in \mathcal{N}$ such that $g \notin S_q^\circ$. This implies that $g \in K_r$ for some $0 < r < q$, and it follows that r divides q, so K_r is a proper subspace of K_q fixed by the action of g. Thus D_g is reducible.

For the reverse direction, assume that D_g is reducible. Then there exists a proper K-subspace V' of K_q which is invariant under Φ. After extension of scalars from K to K_q, Φ is diagonalizable with the diagonal entries of its matrix being $\sigma_i(g) = g(\mu^i z^{1/q})$ for all $0 \leq i < q$, where μ is a primitive q^{th} root of unity. This implies that the diagonal entries of $\Phi|_{V' \otimes K_q} : V' \otimes K_q \to V' \otimes K_q$ will be of the form $\sigma_i(g)$ for some i. However, the Galois group $\mathrm{Gal}(K_q/K)$ acts on K_q, so if the operator $\Phi|_{V' \otimes K_q}$ has one $\sigma_i(g)$ as a diagonal entry, it must have $\sigma_i(g)$ for all i. Thus V' being a proper subspace implies that $\sigma_i(g) = \sigma_j(g)$ for some $0 \leq i \neq j < q$, which only occurs if $g \in K_r$ for some $0 < r < q$.

To prove the second statement, we need to show that if $D = (V, \Phi) \in \mathcal{N}$ is irreducible then $D \simeq D_{g,q}$ for some g and q such that the image of g lies in S_q°. Consider D in canonical form: we extend to $V \otimes K_q$ (assume we choose the minimum such q) and take an eigenvector v of the matrix of $\Phi|_{V \otimes K_q}$ with eigenvalue g. The Galois group $\mathrm{Gal}(K_q/K)$ acts on $V \otimes K_q$, and we let W be the span of the orbit of v in $\mathrm{Gal}(K_q/K)$. Thus W is invariant with respect to the action of the Galois group, so it represents a subspace of V. Moreover, the eigenvalues of the matrix of $\Phi|_{V \otimes K_q}$ are of the form $\sigma_i(g)$, so W is invariant with respect to the action of $\Phi|_{V \otimes K_q}$ as well. Since D is irreducible, W must be all of V and $D \simeq D_{g,q}$. Note that the image of g will be in S_q° because we chose the smallest possible q.

(3) The statement of Theorem 1.2.3 makes it clear that D can be written as a direct sum of indecomposable operators, thus it suffices to prove that for an indecomposable $D \simeq (K_q^m, F_g\varphi)$ we have $D \simeq D_g \otimes T_m$ for some g and m. Specifically, we show that

$$\left(K_q^m, F_g\varphi\right) \simeq \left((K_q, g\varphi) \otimes (K^m, (I_m + \theta N_m)\varphi)\right).$$

From the reduction procedure outlined in [**Pra83**][Lemma 4] we find that

$$\left(K_q^m, \begin{bmatrix} g & & \\ \theta^{1+\lambda} & \ddots & \\ & \ddots & \ddots \end{bmatrix} \varphi\right) \simeq \left(K_q^m, \begin{bmatrix} g & & \\ g\theta & \ddots & \\ & \ddots & \ddots \end{bmatrix} \varphi\right) = \left(K_q^m, g(I_m + \theta N_m)\varphi\right),$$

so all that remains is to show that

$$\left(K_q^m, g(I_m + \theta N_m)\varphi\right) \simeq \left((K_q, g\varphi) \otimes (K^m, (I_m + \theta N_m)\varphi)\right).$$

To give such an isomorphism, we need an invertible map $\phi : K_q^m \to (K_q \otimes K^m)$ such that $\phi^{-1}(g\varphi \otimes (I_m + \theta N_m)\varphi)\phi = g(I_m + \theta N_m)\varphi$. Let $v = \sum v_i e_i \in K_q^m$ where $v_i \in K_q$ and $\{e_i\}$ is the appropriate basis. Let e' and $\{e''_i\}$ be the bases for K_q and K^m respectively. Then we define the map $\phi : K_q^m \to (K_q \otimes K^m)$ on its basis vectors by $\phi(e_i) = e' \otimes e''_i$. Note that ϕ is clearly invertible. We can now show that

$$\begin{aligned}
\phi^{-1}(g\varphi \otimes (I_m + \theta N_m)\varphi)\phi\left(\sum_{i=1}^m v_i e_i\right) &= \phi^{-1}(g\varphi \otimes (I_m + \theta N_m)\varphi)\left(\sum_{i=1}^m v_i e' \otimes e''_i\right) \\
&= \phi^{-1}\left(\sum_{i=1}^m \left[g\varphi(v_i)e' \otimes (e''_i + \theta e''_{i+1})\right]\right) \\
&= \phi^{-1}\left(g\sum_{i=1}^m \varphi(v_i)e' \otimes e''_i + \varphi(v_i)\theta e' \otimes e''_{i+1}\right) \\
&= g\left(\sum_{i=1}^m \varphi(v_i)e_i + \varphi(v_i)\theta e_{i+1}\right) \\
&= g\left(\sum_{i=1}^m \left[\varphi(v_i) + \theta\varphi(v_{i-1})\right] e_i\right) \\
&= g(I_m + \theta N_m)\varphi\left(\sum_{i=1}^m v_i e_i\right).
\end{aligned}$$

Note that in the proof above we use the common notation that $e_{m+1} = v_0 = 0$.

□

1.3. Notation

At times it is useful to keep track of the choice of local coordinate for $\mathcal{C}$ and $\mathcal{N}$, and we denote this with a subscript. To stress the coordinate, we write $\mathcal{C}_0$ to indicate the coordinate z at the point zero, $\mathcal{C}_x$ to indicate the coordinate $z - x := z_x$ at a point $x \neq 0$, and $\mathcal{C}_\infty$ to indicate the coordinate $\zeta = \frac{1}{z}$ at the point at infinity. Note that $\mathcal{C}_0$, $\mathcal{C}_x$ and $\mathcal{C}_\infty$ are all isomorphic to $\mathcal{C}$, but not canonically. Similarly we can write $\mathcal{N}_\infty$ to indicate that we are considering $\mathcal{N}$ with local coordinate at infinity. Since we only work with the point at infinity for $\mathcal{N}$, though, we generally omit the subscript.

We also have a superscript notation for categories, but our conventions for the categories $\mathcal{C}$ and $\mathcal{N}$ are different and a potential source of confusion. Superscript notation for vector spaces with connection is well-established and we continue to use the convention that the superscript corresponds to *slope* (for a formal definition of slope, see [**Kat87**]). Thus, for example, we denote by $\mathcal{C}_\infty^{<1}$ (respectively $\mathcal{C}_\infty^{>1}$) the full subcategory of $\mathcal{C}_\infty$ of connections whose irreducible components all have slopes less than one (respectively greater than one); that is, E_f such that $-1 < \mathrm{ord}(f)$ (respectively $-1 > \mathrm{ord}(f)$).

The correspondence to slope makes sense in the context of connections because all connections have nonnegative slope (i.e. for all E_f we have $\mathrm{ord}(f) \leq 0$). For difference operators we have no such restriction on the order, though, and thus a correspondence to slope would be artificial. The superscripts we use for difference operators therefore refer to the *order* of irreducible components as opposed to the slope. Thus, for example, the notation $\mathcal{N}^{>0}$ indicates the full subcategory of $\mathcal{N}$ of difference operators whose irreducible components D_g have the property that $\mathrm{ord}(g) > 0$.

1.4. The norm and order of an operator

1.4.1. Definition of norm. In the discussion of norms in this subsection we primarily follow the conventions of [**CF86**], though our presentation is self-contained. Fix a real number ϵ such that $0 < \epsilon < 1$. For $f = \sum_{i=k} c_i \theta^{i/q} \in K_q$ with $c_k \neq 0$, we define the order of f as $\mathrm{ord}(f) := k/q$.

Definition 1.4.1. Let $f \in K$. The valuation $|\bullet|$ on K is defined as

$$|f| = \epsilon^{\text{ord}(f)}$$

with $|0| = 0$.

This is a non-archimedean discrete valuation and K is complete with respect to the topology induced by the valuation.

Definition 1.4.2. Let V be a vector space over K. A *non-archimedean norm* on V is a real-valued function $||\bullet||$ on V such that the following hold:

(1) $||v|| > 0$ for $v \in V - \{0\}$.

(2) $||v + w|| \leq \max(||v||, ||w||)$ for all $v, w \in V$.

(3) $||f \cdot v|| = |f| \cdot ||v||$ for $f \in K$ and $v \in V$.

Remark. Norms can be defined more generally, but for our purposes we consider only non-archimedean norms.

Example 1.4.3. The function

$$||(x_1, \ldots, x_n)|| = \max|x_i|$$

is a norm on K^n, and K^n is complete with respect to this norm.

Lemma 1.4.4 ([**CF86**], lemma in Section 2.8)**.** *Any two norms $||\bullet||_1$, $||\bullet||_2$ on a finite-dimensional vector space V over K are equivalent in the following sense: there exists a real number $C > 0$ such that*

$$\frac{1}{C}||\bullet||_1 \leq ||\bullet||_2 \leq C||\bullet||_1.$$

It follows from Lemma 1.4.4 that all norms on a finite-dimensional vector space over K induce the same topology.

Definition 1.4.5. Let $A : V \to V$ be a $\Bbbk$-linear operator. We define the *norm of an operator* to be

$$||A|| = \sup_{v \in V - \{0\}} \left\{ \frac{||A(v)||}{||v||} \right\}.$$

Note that $||A|| < \infty$ if and only if A is continuous ([**KF75**][Chapter 6, Theorem 1]).

1.4.2. Invariant norms. The norm of an operator given in Definition 1.4.5 depends on the choice of the non-archimedean norm $|| \bullet ||$. To find an invariant for norms of operators, consider the following two norms:

Definition 1.4.6. The *infimum* norm is defined as

$$||A||_{inf} = \inf\{||A|| : || \bullet || \text{is a norm on } V\}$$

and the *spectral radius* of A is given by

$$||A||_{spec} = \lim_{n \to \infty} \sqrt[n]{||A^n||}.$$

Note that A must be continuous to guarantee that the limit defining the spectral radius exists. It follows from Lemma 1.4.4 that the spectral radius does not depend on the choice of norm $|| \bullet ||$. For operators in general the spectral radius is often the more useful invariant, but for the class of operators we consider (such as connections, difference operators, and their inverses) the two definitions coincide and we primarily use the infimum norm.

1.4.3. Norms of similitudes.

Proposition 1.4.7. *Let $|| \bullet ||_1$ and $|| \bullet ||_2$ be two norms on V. Then for any invertible $\Bbbk$-linear operator $A : V \to V$, we have $||A||_1 \cdot ||A^{-1}||_2 \geq 1$.*

Proof. Let $v \in V$ and consider the expression

$$\left(\frac{||A^N v||_1}{||v||_1} \right) \left(\frac{||A^N v||_2}{||v||_2} \right)^{-1} \tag{1.3}$$

where $N \gg 0$. By Lemma 1.4.4 we have $\frac{1}{C}||v||_1 \leq ||v||_2 \leq C||v||_1$, and it follows that

$$\frac{1}{C^2} \leq \left(\frac{||A^N v||_1}{||v||_1}\right)\left(\frac{||A^N v||_2}{||v||_2}\right)^{-1} \leq C^2.$$

In particular, (1.3) is bounded below by $\frac{1}{C^2} > 0$ for all N. Suppose that $||A||_1 \cdot ||A^{-1}||_2 < 1$. Then $||A||_1^N \cdot ||A^{-1}||_2^N \to 0$ for $N \gg 0$. We will show that (1.3) cannot exceed $||A||_1^N \cdot ||A^{-1}||_2^N$, which will give us a contradiction. First, for operators A and B we have the property that $||AB|| \leq ||A|| \cdot ||B||$. It follows that

$$||A||_1^N \geq ||A^N||_1 \geq \frac{||A^N v||_1}{||v||_1} \tag{1.4}$$

where the second inequality follows from the definition of norm of an operator. For $w = A^N v$ we also have

$$\begin{aligned}\left(\frac{||A^N v||_2}{||v||_2}\right)^{-1} &= \left(\frac{||w||_2}{||A^{-N}w||_2}\right)^{-1} \\ &= \frac{||A^{-N}w||_2}{||w||_2} \\ &\leq ||A^{-N}||_2 \\ &\leq ||A^{-1}||_2^N.\end{aligned} \tag{1.5}$$

Combining (1.4) and (1.5) it follows that

$$||A||_1^N \cdot ||A^{-1}||_2^N \geq \left(\frac{||A^N v||_1}{||v||_1}\right)\left(\frac{||A^N v||_2}{||v||_2}\right)^{-1}$$

which completes the proof. □

Corollary 1.4.8. *Let $A : V \to V$ be invertible and $||\bullet||$ a norm such that $||A|| \cdot ||A^{-1}|| = 1$. Then $||A|| = ||A||_{inf}$.*

PROOF. Suppose $||\bullet||_1$ is a norm on V such that $||A||_1 < ||A||$. Then $||A||_1 \cdot ||A^{-1}|| < 1$ which contradicts Lemma 1.4.7. □

Definition 1.4.9. Let $||\bullet||$ be a norm on V. Then an operator $A : V \to V$ is a *similitude* (with respect to $||\bullet||$) if $||Av|| = \lambda||v||$ for all $v \in V$. It follows that $||A|| = \lambda$.

Claim 1.4.10. *If $A : V \to V$ is an invertible similitude with $||Av|| = \lambda||v||$, then $||A||_{inf} = \lambda$ and $||A^{-1}|| = \frac{1}{\lambda}$.*

Proof. Since A is a similitude, we have

$$||A^{-1}|| = \sup_{v \in V - \{0\}} \frac{||A^{-1}(Av)||}{||Av||} = \sup_{v \in V - \{0\}} \frac{||v||}{\lambda||v||} = \frac{1}{\lambda}.$$

By Corollary 1.4.8 it follows that $||A|| = ||A||_{inf}$. □

1.4.4. Properties of norms. Given the canonical form of a connection or difference operator, it is quite easy to calculate the norm. With respect to the canonical basis, in particular we note that indecomposable connections with no horizontal sections, indecomposable invertible difference operators, and their inverses are all similitudes.

Remark. We introduce here notation to clear up a potentially confusing situation. The issue is the notation $\nabla = \frac{d}{dz} + A$ for a connection. In particular, at the local coordinate $\zeta = \frac{1}{z}$ the change of variable gives us $\nabla = -\zeta^2 \frac{d}{d\zeta} + A(\zeta)$. To emphasize the local coordinate we will use the notation ∇_z (respectively ∇_ζ) to indicate that we are writing ∇ in terms of z (respectively ζ). In particular we have the equalities $\nabla_z = -\zeta^2 \nabla_\zeta$ and $z\nabla_z = -\zeta\nabla_\zeta$. This change of variable is also discussed in the proof of Claim 3.3.8.

Proposition 1.4.11. *For an indecomposable $(V, \nabla) = (E_f \otimes J_m) \in \mathcal{C}$ such that ∇ has no horizontal sections,*

(1) $||\nabla||_{inf} = \epsilon^{\mathrm{ord}(f)-1}$.

If ∇ is invertible we also have

(2) $||\nabla^{-1}||_{inf} = \epsilon^{-\mathrm{ord}(f)+1}$.

(3) For $(V, \nabla) \in \mathcal{C}_0$, $||(z\nabla)^{-1}||_{inf} = \epsilon^{-\mathrm{ord}(f)}$.

(4) For $(V, \nabla) \in \mathcal{C}_\infty$, $||(z\nabla)^{-1}||_{inf} = ||(\zeta\nabla_\zeta)^{-1}||_{inf} = \epsilon^{-\mathrm{ord}(f)}$.

(5) For $(V, \nabla) \in \mathcal{C}_x$, $||(z\nabla_{z_x})^{-1}||_{inf} = \epsilon^{1-\mathrm{ord}(f)}$.

Proposition 1.4.12. *For an indecomposable $(V, \Phi) = (D_g \otimes T_m) \in \mathcal{N}$,* *(1)* $||\Phi||_{inf} = \epsilon^{\text{ord}(g)}$.

(2) $||(\theta\Phi)^{-1}||_{inf} = \epsilon^{-\text{ord}(g)-1}$.

Proof of Propositions 1.4.11 and 1.4.12. As mentioned above, with respect to the canonical basis and the norm given in Example 1.4.3, it is clear that ∇, $z\nabla$, Φ and $\theta\Phi$ are all similitudes. The results then follow from Claim 1.4.10 and the definitions of E_f and D_g. Note that the result for Proposition 1.4.11, (5) is different from (3) and (4), because in (5) multiplication by z has no effect on the norm of the operator (because the local coordinate is z_x). $\square$

1.4.5. Order of an operator. The *order* of an operator is a notion closely related to the norm of an operator. It is often more convenient to work with order as opposed to norm, so we give a brief introduction to order below.

Definition 1.4.13. Let $B : V \to V$ be a $\Bbbk$-linear operator and $|| \bullet ||$ a norm defined on V. Then the *order* of B is

$$\text{Ord}(B) = \log_\epsilon ||B||_{spec},$$

with $\text{Ord}(0) := \infty$.

Example 1.4.14. The term "order" is suggestive for the following reason. Given Definition 1.4.13, the properties of similitudes, and ∇ an indecomposable connection with no horizontal sections, the following property holds: $\text{Ord}(\nabla) = \ell$ if and only if for all $n \in \mathbb{Q}$ we have $\nabla(z^n I) = (*z^{n+\ell})I+$ higher order terms. Similarly for an indecomposable difference operator Φ, $\text{Ord}(\Phi) = j$ if and only if $\Phi(\theta^n I) = (*\theta^{n+j})I+$ higher order terms. Note that here $* \in \Bbbk - \mathbb{Z}$ if $\ell = -1$ and $* \in \Bbbk$ otherwise.

In the context of the order of an operator, we can state the results of Propositions 1.4.11 and 1.4.12 as follows.

Corollary 1.4.15 (to Propositions 1.4.11 and 1.4.12)**.** *For indecomposable* $(V, \nabla) = (E_f \otimes J_m)$ *in either* $\mathcal{C}_0$ *or* $\mathcal{C}_\infty$ *we have*

(1) $\mathrm{Ord}(\nabla) = \mathrm{ord}(f) - 1$, $\mathrm{Ord}\Big(z\nabla\Big) = \mathrm{ord}(f)$, *and* $\mathrm{Ord}\Big((z\nabla)^{-1}\Big) = -\mathrm{ord}(f)$.

For indecomposable $(V, \nabla_{z_x}) = (E_f \otimes J_m) \in \mathcal{C}_x$,

(2) $\mathit{Ord}\Big(z\nabla_{z_x}\Big) = \mathit{Ord}\Big(\nabla_{z_x}\Big) = \mathrm{ord}(f) - 1$ *and* $\mathrm{Ord}\Big((z\nabla_{z_x})^{-1}\Big) = 1 - \mathrm{ord}(f)$.

For indecomposable $(V, \Phi) = (D_g \otimes U_m) \in \mathcal{N}$

(3) $\mathrm{Ord}(\Phi) = \mathrm{ord}(g)$ *and* $\mathrm{Ord}\Big((\theta\Phi)^{-1}\Big) = -\mathrm{ord}(g) - 1$.

1.5. Operator-root Lemma

In this section we prove a lemma which will be important in the calculation of both the local Fourier and the local Mellin transforms. Namely, the Operator-root Lemma tackles one difficult part of the calculation, that of defining what it means to take the root of a particular operator. To define such a process, we first find a formula for taking integer powers of an operator, and then extend that formula to define fractional powers of the operator as well. The extension of the formula to define a root of the operator relies on techniques found in [**Ari**].

1.5.1. Integer powers of an operator.

Definition 1.5.1. Recall Definition 1.4.13 for the *order* of an operator. Let A and B be $\Bbbk$-linear operators from K_q to K_q. It can be helpful to think of $\mathrm{Ord}(A)$ as

$$\mathrm{Ord}(A) = \inf_{f \in K_q} \left(\mathrm{ord}(Af) - \mathrm{ord}(f)\right), \text{ with } \mathrm{Ord}(0) = \infty.$$

We now define the notation $\underline{o}(z^k)$ by

$$A = B + \underline{o}(z^k) \text{ if and only if } \mathrm{Ord}(A - B) \geq k.$$

Lemma 1.5.2. *Let A and B be $\Bbbk$-linear operators on K_q, with the following conditions: A and $A + B$ are similitudes, and $[A, [B, A]] = 0$. Let $\mathit{Ord}(A) = a$, $\mathit{Ord}(B) = b$, and suppose that $a < b$. Then*

$$(A + B)^m = A^m + mA^{(m-1)}B + \frac{m(m-1)}{2}A^{m-2}[B, A] + \underline{o}(z^{a(m-1)+b}) \tag{1.6}$$

for all $m \in \mathbb{Z}$.

PROOF. We first prove that (1.6) holds for $m \geq 0$ using induction. The case $m = 0$ is trivial. Assuming the equation holds for $(A+B)^m$, we have

$$\begin{aligned}
(A+B)^{m+1} &= (A+B)^m(A+B) \\
&= A^{m+1} + mA^{m-1}BA + \frac{m(m-1)}{2}A^{m-2}[B,A]A + A^mB + \underline{o}(z^{a(m-1)+b+a}) \\
&= A^{m+1} + (m+1)A^mB + mA^{m-1}[B,A] + \frac{m(m-1)}{2}A^{m-1}[B,A] + \underline{o}(z^{am+b}) \\
&= A^{m+1} + (m+1)A^mB + \frac{m(m+1)}{2}A^{m-1}[B,A] + \underline{o}(z^{am+b})
\end{aligned}$$

which completes the induction for the nonnegative integers . Since $A+B$ is invertible, the expansion

$$(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - \dots$$

is well-defined. Using that expansion (which verifies the base case $m = -1$), the proof for $m \leq -1$ follows in the same manner as the proof for the nonnegative integers above. Note that the condition $\mathrm{Ord}(A^{-1}) = -\mathrm{Ord}(A)$ (which follows from A being a similitude) is necessary for the induction on the negative integers. □

1.5.2. Fractional powers of an operator. We now wish to use (1.6) to define fractional powers of the operator $(A+B)$, given certain operators A and B. We follow the method of [**Ari**, Section 7.1] to extend the definition, though our goal is more narrow; Arinkin defines powers for all $\alpha \in \Bbbk$, but we only need to define fractional powers $m \in \frac{1}{p}\mathbb{Z}$ for a given nonzero integer p.

Lemma 1.5.3 (Operator-root Lemma)**.** *Let A and B be the following $\Bbbk$-linear operators on K_q: $A =$ multiplication by $f = jz^{p/q} + \underline{o}(z^{p/q})$, $0 \neq j \in \Bbbk$, and $B = z^n\frac{d}{dz}$ with $n \neq 0$, $p \neq 0$, and $q > 0$ all integers. We have $Ord(A) = \frac{p}{q}$ and $Ord(B) = n - 1$, and we assume* *that* $\frac{p}{q} < n-1$*. Then we can choose a* p^{th} *root of* $(A+B)$*,* $(A+B)^{1/p}$*, such that*

$$(A+B)^m = A^m + mA^{(m-1)}B + \frac{m(m-1)}{2}A^{m-2}[B,A] + \underline{o}(z^{(p/q)(m-1)+n-1})$$

holds for all $m \in \frac{1}{p}\mathbb{Z}$ *where* $(A+B)^m = ((A+B)^{1/p})^{pm}$*.*

Proof. We use the notation found in [**Ari**, Section 7.1]. Letting $P = (1/j)(A+B)$ we have $P : K_q \to K_q$ is $\mathbb{k}$-linear of the form

$$P\left(\sum_\beta c_\beta z^{\beta/q}\right) = \sum_\beta c_\beta \sum_{i\geq 0} p_i(\beta) z^{(\beta+i+p)/q}.$$

Thus $p_0(\beta) = 1$ and all p_i are constants or have the form β/q+constant, so the necessary conditions [**Ari**, Section 7.1, conditions (1) and (2)] are satisfied. We can now define P^m, and likewise $(A+B)^m = j^m P^m$, for $m = \frac{1}{p}$. $\square$

CHAPTER 2

Explicit calculations for local Fourier transforms

In [**BE04**], S. Bloch and H. Esnault introduced the local Fourier transforms for connections on the formal punctured disk. In [**GL04**], R. Garcia Lopez found similar results to [**BE04**] using a different method. Bloch and Esnault also apply the local Fourier transforms to analyze properties of rigidity, in particular proving an invariance result for the rigidity index under the global Fourier transform. Neither [**BE04**] nor [**GL04**] gave explicit calculations for the local Fourier transforms, however. Explicit formulas were proved by J. Fang in [**Fan07**] and C. Sabbah in [**Sab07**]. Interestingly, the calculations rely on different ideas: the proof of [**Fan07**] is more algebraic, while [**Sab07**] uses geometric methods.

In this chapter, we provide yet another proof of the formulas given in [**Fan07**] and [**Sab07**]. Our approach is closer to Fang's, but more straightforward. Specifically, in order to calculate a particular local Fourier transform, one must ascertain the 'canonical form' of the local Fourier transform of a given connection. This amounts to constructing an isomorphism between two connections (on a punctured formal disk). In [**Fan07**], this is done by writing matrices of the connections with respect to certain bases. We work with operators directly, using techniques described by D. Arinkin in [**Ari**, Section 7].

In section 2.1 we define the local Fourier transforms and mention some of their properties. We state theorems for the calculation of the local Fourier transforms in section 2.2, and then prove the theorems in section 2.3. In section 2.4 we show how our results coincide with the earlier computations of Fang in [**Fan07**].

2.1. Local Fourier transforms

Recall the following notation: we write $\mathcal{C}_0$ to indicate the category $\mathcal{C}$ with the coordinate z at the point zero and $\mathcal{C}_\infty$ to indicate the coordinate $\zeta = \frac{1}{z}$ at the point at infinity. We also denote by $\mathcal{C}_\infty^{<1}$ (respectively $\mathcal{C}_\infty^{>1}$) the full subcategory of $\mathcal{C}_\infty$ of connections whose irreducible components all have slope less than one (respectively greater than one); that is, E_f such that $-1 < \mathrm{ord}(f)$ (respectively $-1 > \mathrm{ord}(f)$). Note that for $0 \neq x \in \Bbbk$, calculation of the local Fourier transform can be reduced to the $\mathcal{C}_0$ case, so we do not give explicit formulas for objects in $\mathcal{C}_x$.

Definition 2.1.1. We define the local Fourier transforms $\mathcal{F}^{(0,\infty)}$, $\mathcal{F}^{(\infty,0)}$ and $\mathcal{F}^{(\infty,\infty)}$ using the relations given in [**BE04**, Propositions 3.7, 3.9 and 3.12] while following the convention of [**Ari**, Section 2.2]. We let the Fourier transform coordinate of z be $\hat{z}$, with $\hat{\zeta} = \frac{1}{\hat{z}}$. Let $E = (V, \nabla_z) \in \mathcal{C}_0$ such that ∇_z has no horizontal sections, thus ∇_z is invertible. The following is a precise definition for $\mathcal{F}^{(0,\infty)}(E)$, the other local Fourier transforms can be defined analogously and thus precise definitions are omitted. Consider on V the $\Bbbk$-linear operators

$$\hat{\zeta} = -\nabla_z^{-1} : V \to V \text{ and } \hat{\nabla}_{\hat{\zeta}} = -\hat{\zeta}^{-2} z : V \to V. \tag{2.1}$$

As in [**Ari**], $\hat{\zeta}$ extends to define an action of $\Bbbk((\hat{\zeta}))$ on V and $\dim_{\Bbbk((\hat{\zeta}))} V < \infty$. We write $V_{\hat{\zeta}}$ to indicate that we are considering V as a $\Bbbk((\hat{\zeta}))$-vector space. Then $\hat{\nabla}_{\hat{\zeta}}$ is a connection, and the $\Bbbk((\hat{\zeta}))$-vector space $V_{\hat{\zeta}}$ with connection $\hat{\nabla}_{\hat{\zeta}}$ is denoted by

$$\mathcal{F}^{(0,\infty)}(E) := (V_{\hat{\zeta}}, \hat{\nabla}_{\hat{\zeta}}) \in \mathcal{C}_\infty^{<1},$$

which defines the functor $\mathcal{F}^{(0,\infty)} : \mathcal{C}_0 \to \mathcal{C}_\infty^{<1}$.

Given the conventions above, we can express the other local Fourier transforms by the functors

$$\mathcal{F}^{(\infty,0)} : \mathcal{C}_\infty^{<1} \to \mathcal{C}_0 \text{ and } \mathcal{F}^{(\infty,\infty)} : \mathcal{C}_\infty^{>1} \to \mathcal{C}_\infty^{>1}.$$

If one considers only the full subcategories of $\mathcal{C}_0$ and $\mathcal{C}_\infty^{<1}$ of connections with no horizontal sections, the functors $\mathcal{F}^{(0,\infty)}$ and $\mathcal{F}^{(\infty,0)}$ define an equivalence of categories. Similarly, $\mathcal{F}^{(\infty,\infty)}$ is an auto-equivalence of the subcategory $\mathcal{C}_\infty^{>1}$ [**BE04**, Propositions 3.10 and 3.12].

2.2. Statement of theorems

Let s be a nonnegative integer and r a positive integer.

2.2.1. Calculation of $\mathcal{F}^{(0,\infty)}$.

Theorem 2.2.1. *Let $f \in R_r^\circ(z)$ with ord$(f) = -s/r$ and $f \neq 0$. Then $E_f \in \mathcal{C}_0$ and*

$$\mathcal{F}^{(0,\infty)}(E_f) \simeq E_g,$$

where $g \in R_{r+s}^\circ(\hat{\zeta})$ is determined by the following system of equations:

$$f = -z\hat{z} \tag{2.2}$$

$$g = f + \frac{s}{2(r+s)} \tag{2.3}$$

Remark. Recall that $\hat{\zeta} = \frac{1}{\hat{z}}$. We determine g using (2.2) and (2.3) as follows. First, using (2.2) we express z in terms of $\hat{\zeta}^{1/(r+s)}$. We then substitute that expression for z into (2.3) and solve to get an expression for $g(\hat{\zeta})$ in terms of $\hat{\zeta}^{1/(r+s)}$.

When we use (2.2) to write an expression for z in terms of $\hat{\zeta}^{1/(r+s)}$, the expression is not unique since we must make a choice of a root of unity. More concretely, let η be a primitive $(r+s)^{\text{th}}$ root of unity. Then replacing $\hat{\zeta}^{1/(r+s)}$ with $\eta\hat{\zeta}^{1/(r+s)}$ in our equation for z will yield another possible expression for z. This choice will not affect the overall result, however, since all such expressions will lie in the same Galois orbit. Thus by Proposition 1.1.4 (1), they all correspond to the same connection.

Corollary 2.2.2. *Let E be an object in $\mathcal{C}$. By Proposition 1.1.4 (3), let E have decomposition $E \simeq \bigoplus_i \left(E_{f_i} \otimes J_{m_i}\right)$. Then*

$$\mathcal{F}^{(0,\infty)}(E) \simeq \bigoplus_i \left(E_{g_i} \otimes J_{m_i}\right)$$

for $E_{g_i} = \mathcal{F}^{(0,\infty)}(E_{f_i})$ as defined in Theorem 2.2.1.

Sketch of Proof. $E_f \otimes J_m$ is the unique indecomposable object in $\mathcal{C}$ formed by m successive extensions of E_f. Since we have an equivalence of categories, we only need to know how $\mathcal{F}^{(0,\infty)}$ acts on E_f. This is given by Theorem 2.2.1. $\square$

2.2.2. Calculation of $\mathcal{F}^{(\infty,0)}$.

Theorem 2.2.3. *Let $f \in R_r^\circ(\zeta)$ with ord$(f) = -s/r$, $s < r$, and $f \neq 0$. Then $E_f \in \mathcal{C}_\infty^{<1}$ and*

$$\mathcal{F}^{(\infty,0)}(E_f) \simeq E_g,$$

where $g \in R_{r-s}^\circ(\hat{z})$ is determined by the following system of equations:

$$f = z\hat{z} \tag{2.4}$$

$$g = -f + \frac{s}{2(r-s)} \tag{2.5}$$

Remark. We determine g from (2.4) and (2.5) as follows. First, we use (2.4) to express ζ in terms of $\hat{z}^{1/(r-s)}$. We then substitute this expression into (2.5) to get an expression for $g(\hat{z})$ in terms of $\hat{z}^{1/(r-s)}$.

2.2.3. Calculation of $\mathcal{F}^{(\infty,\infty)}$.

Theorem 2.2.4. *Let $f \in R_r^\circ(\zeta)$ with ord$(f) = -s/r$ and $s > r$. Then $E_f \in \mathcal{C}_\infty^{>1}$ and*

$$\mathcal{F}^{(\infty,\infty)}(E_f) \simeq E_g,$$

where $g \in R^{\circ}_{s-r}(\hat{\zeta})$ is determined by the following system of equations:

$$f = z\hat{z} \tag{2.6}$$

$$g = -f + \frac{s}{2(s-r)} \tag{2.7}$$

Remark. We determine g from (2.6) and (2.7) as follows. First, we use (2.6) to express ζ in terms of $\hat{\zeta}^{1/(s-r)}$. We then substitute this expression into (2.7) to get an expression for $g(\hat{\zeta})$ in terms of $\hat{\zeta}^{1/(s-r)}$.

2.3. Proof of theorems

2.3.1. Outline of proof of Theorem 2.2.1. We start with the operators given in (2.1), viewing them as equivalent operators on K_r. We wish to understand how the operator $\hat{\nabla}_{\hat{\zeta}}$ acts in terms of the operator $\hat{\zeta}$. The proof is broken into two cases, depending on the type of singularity. In the case of regular singularity, we have $\operatorname{ord}(f) = 0$, and the proof is fairly straightforward. In the irregular singularity case where $\operatorname{ord}(f) < 0$, the proof hinges upon defining a fractional power of an operator. This was done in Lemma 1.5.3, the Operator-root Lemma. Lemma 1.5.3 is the heavy lifting of the proof; the remaining portion is just calculation to extract the appropriate constant term (see remark below) from the expression given by Lemma 1.5.3.

Remark. We give a brief explanation regarding the origin of the system of equations found in Theorem 2.2.1. Consider the expressions given in (2.1). Suppose we were to make a "naive" local Fourier transform over K_r by defining $\nabla_z = z^{-1}f(z)$ and $\hat{\nabla}_{\hat{\zeta}} = \hat{\zeta}^{-1}g(\hat{\zeta})$; in other words, as in Definition 1.1.1 but without the differential parts. Then from the equation $-(z^{-1}f)^{-1} = \hat{\zeta}$ we conclude

$$f = -z\hat{z}. \tag{2.8}$$

Similarly, from $-\hat{\zeta}^{-2}z = \hat{\zeta}^{-1}g$ we find $-\hat{z}z = g$, which when combined with (2.8) gives

$$f = g. \tag{2.9}$$

When one incorporates the differential parts into the expressions for ∇_z and $\hat{\nabla}_{\hat{\zeta}}$, one sees that the system of equations (2.8) and (2.9) nearly suffices to find the correct expression for $g(\hat{\zeta})$, only a constant term is missing. This constant term arises from the interplay between the differential and linear parts of ∇_z, and we wish to derive what the value of it is. Similar calculations can be carried out to justify the systems of equations for Theorems 2.2.3 and 2.2.4.

2.3.2. Proof of Theorem 2.2.1.

Proof. From [**BE04**, Proposition 3.7] we have the following equations for the local Fourier transform $\mathcal{F}^{(0,\infty)}$:

$$z = -\hat{\zeta}^2(\partial_{\hat{\zeta}}) \text{ and } \partial_z = -\hat{\zeta}^{-1}. \tag{2.10}$$

Converting to our notation, we write $\partial_{\hat{\zeta}} = \hat{\nabla}_{\hat{\zeta}} = \frac{d}{d\hat{\zeta}} + \hat{\zeta}^{-1}g(\hat{\zeta})$ and $\partial_z = \nabla_z = \frac{d}{dz} + z^{-1}f(z)$. Then (2.10) becomes

$$z = -\hat{\zeta}^2\frac{d}{d\hat{\zeta}} - \hat{\zeta}g(\hat{\zeta}) \tag{2.11}$$

and

$$\frac{d}{dz} + z^{-1}f(z) = -\hat{\zeta}^{-1}. \tag{2.12}$$

Our goal is to use (2.12) to write an expression for the operator z in terms of $\hat{\zeta}$, at which point we can substitute into (2.11) to find an expression for $g(\hat{\zeta})$.

Case One: Regular singularity ($\text{ord}(f) = 0$).

In this case we have $s = 0$ and $r = 1$, so $f = \alpha \in \Bbbk - \mathbb{Z}$. Then (2.12) has form $\frac{d}{dz} + \frac{\alpha}{z} = -\hat{\zeta}^{-1}$. But on K, the operator $\frac{d}{dz}$ acts on monomials as multiplication by $\frac{n}{z}$ for some $n \in \mathbb{Z}$, and $f \in R_r^{\circ}(z)$ means that α is only defined up to a shift by $\mathbb{Z}$. Thus the operator $\frac{d}{dz} + \frac{\alpha}{z}$ acts in the same manner as just $\frac{\alpha}{z}$. In other words, we can safely ignore the differential part of the operator in the case of a regular singularity. The remainder of this case follows from the remark below the outline in subsection 2.3.1.

Case Two: Irregular singularity ($\operatorname{ord}(f) < 0$).

Consider the equation

$$z^{-1} f = -\hat{\zeta}^{-1}, \tag{2.13}$$

which is (2.12) without the differential part, and coincides with (2.2). Equation (2.13) can be thought of as an implicit expression for the variable z in terms of $\hat{\zeta}$, which one can rewrite as an explicit expression $z = h(\hat{\zeta}) \in \mathbb{k}((\hat{\zeta}^{1/(r+s)}))$ for the variable z. This is the purely algebraic calculation which in Theorem 2.2.1 is stated as expressing z in terms of $\hat{\zeta}^{1/(r+s)}$. Note that since there is no differential part in (2.13), $h(\hat{\zeta})$ is <u>not</u> the same as the operator z. Since the leading term of $z^{-1}f(z)$ is $az^{-(r+s)/r}$ (for some $a \in \mathbb{k}$), (2.13) implies that $h(\hat{\zeta}) = a^{r/(r+s)}(-\hat{\zeta})^{r/(r+s)} + \underline{o}(\hat{\zeta}^{r/(r+s)})$. Using (2.12) we find that the operator z will be of the form

$$z = h(\hat{\zeta}) + *(-\hat{\zeta}) + \underline{o}(\hat{\zeta}) \tag{2.14}$$

where the $* \in \mathbb{k}$ represents the coefficient that arises from the interplay between the differential and linear parts of $(-\hat{\zeta}) = (\nabla_z)^{-1}$. As explained in the outline, we wish to find the value of *. Let $A = z^{-1}f(z)$ and $B = \frac{d}{dz}$, then $[B, A] = A' = z^{-1}f' - z^{-2}f$. From (2.12) we have $-\hat{\zeta} = (A+B)^{-1}$, and we apply the Operator-root Lemma (Lemma 1.5.3) to find

$$(-\hat{\zeta})^{\frac{r}{r+s}} = a^{\frac{-r}{r+s}} \left(z + \cdots + a^{-1} \left[\frac{-r}{r+s} \left(\frac{\mathbb{Z}}{r} \right) + \frac{-r}{r+s} + \frac{-s}{2(r+s)} \right] z^{\frac{r+s}{r})} + \underline{o}(z^{\frac{r+s}{r}}) \right).$$

Remark. We use the notation $\frac{\mathbb{Z}}{r}$ to represent the operator $z\frac{d}{dz}$. This notation makes sense, because $z\frac{d}{dz} : K_r \to K_r$ acts as $z\frac{d}{dz}(z^{n/r}) = \frac{n}{r}(z^{n/r})$ for any $n \in \mathbb{Z}$.

Also from Lemma 1.5.2 we have

$$(-\hat{\zeta}) = a^{-1}z^{1+(s/r)} + \underline{o}(z^{1+(s/r)}).$$

The appropriate value for * in (2.14) is the expression that will make the leading term of $*(-\hat{\zeta})$, which will be $*a^{-1}z^{1+(s/r)}$, cancel with $a^{-1}\left[\frac{-\mathbb{Z}}{r+s} + \frac{-r}{r+s} + \frac{-s}{2(r+s)}\right] z^{1+(s/r)}$. Thus we find that

$$* = \frac{\mathbb{Z}+r}{r+s} + \frac{s}{2(s+r)}. \tag{2.15}$$

Applying both sides of (2.11) to $1 \in K_r$, and using the fact that $\frac{d}{d\hat{\zeta}}(1) = 0$, we see that $z = -\hat{\zeta}g(\hat{\zeta})$. Thus to find the expression for g we simply need to compute the Laurent series in $\hat{\zeta}$ given by $(-\hat{\zeta}^{-1})z$. Substituting the expressions from (2.14) and (2.15) into $(-\hat{\zeta}^{-1})z$, we have

$$g(\hat{\zeta}) = -\hat{\zeta}^{-1}h(\hat{\zeta}) + \left(\frac{\mathbb{Z}+r}{r+s} + \frac{s}{2(r+s)}\right) + \underline{o}(1).$$

By Proposition 1.1.4, (1), $E_{g,r+s}$ will be isomorphic to $E_{\dot{g},r+s}$ where

$$\dot{g}(\hat{\zeta}) = -\hat{\zeta}^{-1}h(\hat{\zeta}) + \frac{s}{2(r+s)}, \tag{2.16}$$

since g and $\dot{g}$ differ only by $\frac{\mathbb{Z}+r}{r+s} \in \frac{1}{r+s}\mathbb{Z}$. From (2.13) we have $-\hat{\zeta}^{-1}h(\hat{\zeta}) = -z\hat{z} = f$, so (2.16) matches (2.3) which completes the proof. □

2.3.3. Proof of Theorem 2.2.3.

Proof. This proof is much the same as the proof of Theorem 2.2.1, so we only sketch the pertinent details. From [**BE04**, Proposition 3.9], in our notation we have

$$\zeta^2\nabla_\zeta = \hat{z} \text{ and } \zeta^{-1} = -\hat{\nabla}_{\hat{z}} \tag{2.17}$$

We wish to write $z = \zeta^{-1}$ in terms of $\hat{z}^{1/(r-s)}$. Consider the equation

$$\zeta f = \hat{z} \tag{2.18}$$

which is the first equation of (2.17) without the differential part. We can think of (2.18) as an implicit definition for the variable ζ, which we can rewrite as an explicit expression $\zeta = h(\hat{z}) = a^{-r/(r-s)}\hat{z}^{r/(r-s)} + \underline{o}(\hat{z}^{r/(r-s)})$. Letting $A = \zeta f(\zeta)$, $B = \zeta^2 \frac{d}{d\zeta}$ and $\hat{z} = A + B$, we have $[B, A] = \zeta^2 A'$ and the Operator-root Lemma gives

$$\hat{z}^{r/(r-s)} = a^{r/(r-s)} \left(\zeta + \cdots + a^{-1} \left[\frac{r}{r-s} \left(\frac{\mathbb{Z}}{r} \right) + \frac{s}{2(r-s)} \right] \zeta^{1+(s/r)} + \underline{o}(\zeta^{1+(s/r)}) \right)$$

and

$$\hat{z}^{(r+s)/(r-s)} = a^{(r+s)/(r-s)} \zeta^{1+(s/r)} + \underline{o}(\zeta^{1+(s/r)}).$$

We conclude that the operator ζ will be

$$\zeta = h(\hat{z}) + a^{-2r/(r-s)} \left[\frac{-\mathbb{Z}}{r-s} + \frac{-s}{2(r-s)} \right] \hat{z}^{(r+s)/(r-s)} + \underline{o}(\hat{z}^{(r+s)/(r-s)}).$$

Inverting the operator ζ, we find

$$\zeta^{-1} = z = h(\hat{z})^{-1} + \left(\frac{\mathbb{Z}}{r-s} + \frac{s}{2(r-s)} \right) \hat{z}^{-1} + \underline{o}(\hat{z}^{-1})$$

and it follows that

$$g(\hat{z}) = -\hat{z}z = -\hat{z}h(\hat{z})^{-1} + \frac{-\mathbb{Z}}{r-s} + \frac{-s}{2(r-s)} + \underline{o}(1).$$

Note that $f = \hat{z}h(\hat{z})^{-1}$. As in the proof of Theorem 2.2.1, we use Proposition 1.1.4, (1), to find an object isomorphic to E_g which matches the object given in the theorem, completing the proof of Theorem 2.2.3. $\square$

2.3.4. Proof of Theorem 2.2.4.

PROOF. The calculations are virtually identical to the proof of Theorem 2.2.3, but the expressions are written in terms of $\hat{\zeta}$ instead of $\hat{z}$, and $s - r$ instead of $r - s$. Starting with [**BE04**, Proposition 3.12], in our notation we have

$$\zeta^2 \nabla_\zeta = \hat{z} \text{ and } \zeta^{-1} = -\hat{\zeta}^2 \hat{\nabla}_{\hat{\zeta}}.$$

Repeating the calculations of Theorem 2.2.3 we conclude that

$$g(\hat{\zeta}) = -\hat{\zeta}^{-1}z = -\hat{\zeta}^{-1}h(\hat{\zeta})^{-1} + \frac{\mathbb{Z}}{s-r} + \frac{s}{2(s-r)} + \underline{o}(1).$$

Note that $-\hat{\zeta}^{-1}h(\hat{\zeta})^{-1} = f$. As before, by considering an appropriate isomorphic object we eliminate the term with $\mathbb{Z}$, completing the proof of Theorem 2.2.4. $\square$

2.4. Comparison with previous results

One notes that in [**Fan07**], Fang's Theorems 1, 2, and 3 look slightly different from those given in (respectively) our Theorems 2.2.1, 2.2.3, and 2.2.4. We shall present a brief explanation for the equivalence of Fang's Theorem 1 and our Theorem 2.2.1. One large difference in our methods is that Fang's calculations are split into a regular and irregular part, whereas we calculate both parts simultaneously. We first verify the equivalence for the irregular part.

2.4.1. Equivalence for the irregular part. Suppose f in Theorem 2.2.1 has zero regular part. In particular, this means that f has no constant term. Then with Fang's notation on the left and our notation on the right, we have the following relationships:

$$t \text{ corresponds to } z$$

$$t' \text{ corresponds to } \hat{z}$$

$$t\partial_t(\alpha) \text{ corresponds to } f$$

$$(1/t')\partial_{(1/t')}(\beta) + \frac{s}{2(r+s)} \text{ corresponds to } g$$

Using the correspondences above and equation (2.1) from Fang's paper, one can manipulate the systems of equations to see that the theorems coincide on the irregular part.

2.4.2. Equivalence for the regular part. In [**Fan07**], the structure of the theorems is such that the calculation of the regular part is quite straightforward. Using our theorems, however, the calculation of the regular part is hidden. To verify that the regular portion of our calculation matches up with the results from [**Fan07**], it suffices to prove the claim below. We note that one can also calculate the regular part by using the global Fourier transform and meromorphic Katz extension; our proof is independent of that method.

Claim 2.4.1. *Let* $f(z) = az^{-s/r} + \cdots + b$ *as in Theorem 2.2.1 and* $\mathcal{F}^{(0,\infty)}(E_f) = E_g$. *Then* g *will have constant term* $\left(\frac{r}{r+s}\right)b + \frac{s}{2(r+s)}$.

Before we prove Claim 2.4.1, we first prove two lemmas regarding general facts about formal Laurent series and compositional inverses.

Lemma 2.4.2. *Let* $j(z) \in K_q$ *with ord*$(j) = \frac{p}{q}$, $p \in \mathbb{Z} - \{0\}$ *and* $q > 0$. *If* $p > 0$, *then* j *has a formal compositional inverse* $j^{\langle -1 \rangle} \in \Bbbk((z^{1/p}))$. *If* $p < 0$, *then* j *has a formal compositional inverse* $j^{\langle -1 \rangle} \in \Bbbk((\zeta^{1/p}))$.

Proof. Let $h(z) = (z^{1/p} \circ j \circ z^q)(z)$. Then $h(z)$ is a formal power series with no constant term and a nonzero coefficient for the z term. Such a power series will have a compositional inverse, call it $h^{\langle -1 \rangle}(z)$. Then $j^{\langle -1 \rangle}(z) := (z^q \circ h^{\langle -1 \rangle} \circ z^{1/p})(z)$ will be a compositional inverse for j. □

Remark. Note that h (and $h^{\langle -1 \rangle}$ as well) is not unique since a choice of root of unity is made. This will not affect our result, though, since h^p and $(h^{\langle -1 \rangle})^q$ will be unique.

Lemma 2.4.3. *Let* $j(z) = az^{-(r+s)/r} + \cdots + bz^{-1} + \underline{o}(z^{-1})$, $j(z) \in K_r$, *with* s *a non-negative integer and* $r \in \mathbb{Z}^+$. *Then the coefficient for the* z^{-1} *term of* $j^{\langle -1 \rangle}(z)$ *will be* $\frac{br}{r+s}$.

Proof. Let $h(z) = (z^{-1/(r+s)} \circ j \circ z^r)(z)$. Then $j(z^r) = h^{-(r+s)}$ and from the proof of Lemma 2.4.2 we have

$$j^{\langle -1 \rangle}(z^{-(r+s)}) = (h^{\langle -1 \rangle})^r. \tag{2.19}$$

According to the Lagrange inversion formula, the coefficients of h and $h^{\langle -1\rangle}$ are related by

$$(r+s)[z^{r+s}](h^{\langle -1\rangle})^r = r[z^{-r}]h^{-(r+s)} \tag{2.20}$$

where $[z^{r+s}](h^{\langle -1\rangle})^r$ denotes the coefficient of the z^{r+s} term in the expansion of $(h^{\langle -1\rangle})^r$. Substituting (2.19) and $j(z^r) = h^{-(r+s)}$ into (2.20) we conclude that

$$[z^{r+s}]j^{\langle -1\rangle}(z^{-(r+s)}) = \frac{r}{r+s}[z^{-r}]j(z^r) \tag{2.21}$$

Since $[z^{-r}]j(z^r) = b$, the conclusion follows. $\square$

PROOF OF CLAIM 2.4.1. Given the notation used above for the Lagrange inversion formula, we can restate the claim as follows: if $[z^0]f = b$, then $[\hat{\zeta}^0]g = \frac{br}{r+s} + \frac{s}{2(r+s)}$. Let $j(z) = -z^{-1}f$. Then

$$[z^{-1}]j = -[z^0]f = -b.$$

By (2.2) we conclude that $\hat{z} = j(z)$, and by Lemma 2.4.2 let $j^{\langle -1\rangle}$ be the compositional inverse. Then $j^{\langle -1\rangle}(\hat{z}) = z$. From (2.3) we have $g = -z\hat{z} + \frac{s}{2(r+s)}$, which implies that $-\hat{z}^{-1}(g - \frac{s}{2(r+s)}) = j^{\langle -1\rangle}(\hat{z})$. This gives

$$[\hat{z}^{-1}]j^{\langle -1\rangle} = -[\hat{z}^0]g + \frac{s}{2(r+s)}$$

or equivalently

$$[\hat{z}^0]g = -[\hat{z}^{-1}]j^{\langle -1\rangle} + \frac{s}{2(r+s)}. \tag{2.22}$$

By Lemma 2.4.3, $[z^{-1}]j = -b$ implies that $[\hat{z}^{-1}]j^{\langle -1\rangle} = \frac{-br}{r+s}$. The result then follows from (2.22) and noting that $[\hat{z}^0]g = [\hat{\zeta}^0]g$. $\square$

CHAPTER 3

Introduction of local Mellin transforms

In this chapter we give rigorous definitions for the local Mellin transforms for connections on the punctured formal disk, as well as define the local inverse Mellin transforms. We also prove that the local Mellin transforms induce certain equivalences of categories, but leave explicit calculations for the next chapter. We denote the local Mellin transforms by $\mathcal{M}^{(0,\infty)}$, $\mathcal{M}^{(x,\infty)}$, and $\mathcal{M}^{(\infty,\infty)}$, where the superscripts are indicative of the point of singularity in the same fashion as for the local Fourier transforms. The results and definitions of this chapter are analogous to those for the local Fourier transforms found in Chapter 2, section 2.1, as well as [**Ari**], [**BE04**] and [**GL04**].

In section 3.1 we present definitions and results regarding Tate vector spaces (which are also found in [**Ari**, Section 5.3]) that are pertinent to our construction of the local Mellin transform. Section 3.2 is devoted to taking the results in section 3.1 and aligning them to our particular situation. In sections 3.3 and 3.4 we give formal definitions of the local Mellin and local inverse Mellin transforms, as well as prove that they are well-defined. We demonstrate the equivalences of categories given by the local Mellin transforms in section 3.5.

3.1. Definitions and previous results

3.1.1. The z-adic topology. Let $A = \Bbbk[[z]]$.

Definition 3.1.1. We define the z-adic topology on the vector space V as follows: a *lattice* is a $\Bbbk$-subspace $L \subset V$ that is of the form $L = \bigoplus_i Ae_i$ for some basis e_i of V over K. Then the *z-adic topology on V* is defined by letting the basis of open neighborhoods of $v \in V$ be cosets $v + L$ for all lattices $L \subset V$.

Note that one can fix a basis $\{e_i\}$ and define certain lattices $L_k = \bigoplus_i z^k A e_i$ for all $k \in \mathbb{Z}$. Then for any open neighborhood $v + L$ of $v \in V$, there exists $k \in \mathbb{Z}$ such that $v + L_k \subset v + L$. In particular, let $\{e_i'\}$ be the basis corresponding to L and $C = (c_{ij})$ be the change of basis matrix sending e_i to e_i'. Then $k = -\min\{\mathrm{ord}(c_{ij})\}$ will do.

Remark. An equivalent definition for the z-adic topology, without reference to choice of basis, is given in [**Ari**, Section 4.2]. The z-adic topology is also equivalent to the topology induced by any norm, as described in Lemma 1.4.4.

For ease of explication, we copy the remaining definitions and results in this section from [**Ari**, Section 5.3]. For more details on Tate vector spaces, see [**BD04**, Section 2.7.7].

3.1.2. Tate vector spaces.

Definition 3.1.2. Let V be a topological vector space over $\Bbbk$, where $\Bbbk$ is equipped with the discrete topology. V is *linearly compact* if it is complete, Hausdorff, and has a base of neighborhoods of zero consisting of subspaces of finite codimension. Equivalently, a linearly compact space is the topological dual of a discrete space.
V is a *Tate space* if it has a linearly compact open subspace.

Definition 3.1.3. An A-module M is *of Tate type* if there is a finitely generated submodule $M' \subset M$ such that M/M' is a torsion module that is 'cofinitely generated' in the sense that

$$\dim_{\Bbbk} \mathrm{Ann}_z(M/M') < \infty, \text{ where } \mathrm{Ann}_z(M/M') = \{m \in M/M' | zm = 0\}.$$

Lemma 3.1.4.

(1) Any finitely generated A-module M is linearly compact in the z-adic topology.

(2) Any A-module of Tate type is a Tate vector space in the z-adic topology.

Proposition 3.1.5. *Let V be a Tate space. Suppose an operator $Z : V \to V$ satisfies the following conditions:*

(1) Z is continuous, open and (linearly) compact. In other words, if $V' \subset V$ is an open linearly compact subspace, then so are $Z(V')$ and $Z^{-1}(V')$.

(2) Z is contracting. In other words, $Z^n \to 0$ in the sense that for any linearly compact subspace $V' \subset V$ and any open subspace $U \subset V$, we have $Z^n(V') \subset U$ for $n \gg 0$.

Then there exists a unique structure of a Tate type A-module on V such that $z \in A$ acts as Z and the topology on V coincides with the z-adic topology.

Definition 3.1.6. For a Tate space V, if an operator $Z : V \to V$ satisfies the conditions of Proposition 3.1.5, we say it is *nicely contracting*. If an operator Z is invertible and Z^{-1} satisfies the conditions of Proposition 3.1.5, it is *nicely expanding*.

3.2. Lemmas

We prove some lemmas describing our situation in the language of Tate vector spaces.

Lemma 3.2.1. *Let $Z : V \to V$ be a $\Bbbk$-linear operator. If Ord$(Z) > 0$, then Z is contracting.*

Proof. Let $\{e_i\}$ be a basis over which Z has positive order and $k \in \mathbb{Z}$. It suffices to show that for the lattice $L = \bigoplus_i z^k A e_i$, any open subspace U, and some $N \in \mathbb{Z}$, we have $Z^n(L) \subset U$ for all $n \geq N$. The fact that U is an open subspace implies that $L' \subset U$ where L' is the lattice $\bigoplus_i z^j A e_i$ for some $j \in \mathbb{Z}$. We show that $Z^n(L) \subset L'$. Because $\mathrm{Ord}(Z) = u > 0$ we will have $Z(L) \subset \bigoplus_i z^{k+u} A e_i$ as well as $Z^n(L) \subset \bigoplus_i z^{k+nu} A e_i$. As n increases, the exponent $k + nu$ can be made arbitrarily large, in particular larger than j. We conclude that $Z^n(L) \subset L' \subset U$ for $n \gg 0$. $\square$

Lemma 3.2.2. *A K-vector space V is of Tate-type if and only if it is finite dimensional.*

Proof. Note that as a K-vector space, V also has the structure of an A-module. For the forward direction, there exists a finitely generated A-submodule $M' \subset V$ such that

V/M' is a torsion A-module. It follows that for every $v \in V$ there exists $a \in A - \{0\}$ such that $av \in M'$. Then $av = \sum^{n} a_i m'_i$ where $a_i \in A$ and $\{m'_i\}$ are the generators of M', so $v = \sum^{n} (a_i/a) m'_i$ and thus V is generated by $\{m'_i\}$. Note that $\frac{a_i}{a} \in K$, thus the condition that V be a vector space is necessary.

For the reverse implication, let $V \simeq K^n$ with basis $\{e_i\}$. Then $M' = \bigoplus_i Ae_i$ is a finitely generated A-submodule such that V/M' is a torsion module. Since

$$\dim_{\Bbbk} \mathrm{Ann}_z(V/M') = n,$$

V/M' is also cofinitely generated. Thus V is of Tate-type. □

Lemma 3.2.3. *For any object $(V, \nabla) \in \mathcal{C}$ or $(V, \Phi) \in \mathcal{N}$, V is a Tate vector space.*

Proof. Since V is finite dimensional, this follows from Lemma 3.2.2 and Lemma 3.1.4,(2). □

3.3. Definition of local Mellin transforms

Below we give definitions of the local Mellin transforms. To alleviate potential confusion, let us explain the format we will use for the definitions. We begin by stating the definition in its entirety, but it is not *a priori* clear that all statements of the definition are true. We then claim that the transform is in fact well-defined and give a proof to clear up the questionable parts of the definition.

3.3.1. Definition of $\mathcal{M}^{(0,\infty)}$.

Definition 3.3.1. Let $E = (V, \nabla) \in \mathcal{C}_0^{>0}$. Thus all indecomposable components of ∇ have slope greater than zero, so for each indecomposable component $E_f \otimes J_m$ one has $\mathrm{ord}(f) < 0$. Consider on V the $\Bbbk$-linear operators

$$\theta := -(z\nabla)^{-1} : V \to V \text{ and } \Phi := z : V \to V \tag{3.1}$$

Then θ extends to an action of $\Bbbk((\theta))$ on V, $\dim_{\Bbbk((\theta))} V < \infty$, and Φ is an invertible difference operator. We write $V = V_\theta$ to denote that we are considering V as a $\Bbbk((\theta))$-vector space. We define *the local Mellin transform from zero to infinity of E* to be the object

$$\mathcal{M}^{(0,\infty)}(E) := (V_\theta, \Phi) \in \mathcal{N}.$$

Claim 3.3.2. *$\mathcal{M}^{(0,\infty)}$ is well-defined.*

Proof. To prove the claim we must show the following:

(i) θ extends to an action of $\Bbbk((\theta))$ on V.

(ii) V_θ is finite dimensional.

(iii) Φ is an invertible difference operator on V_θ.

We prove (i) with Lemma 3.3.3 below. In the proof of Lemma 3.3.3 we show that $z\nabla$ is nicely expanding, and it follows by Proposition 3.1.5 that V_θ is of Tate type. Lemma 3.2.2 then implies that V_θ is finite-dimensional, proving (ii). To prove (iii), we first note that Φ is invertible by construction. To see that Φ is a difference operator, we first show that it suffices to show that for $\eta = \theta^{-1}$ one has $\Phi\eta = (\eta+1)\Phi$. By definition, we need to show that $\Phi(fv) = \varphi(f)\Phi(v)$ for all $f \in K$ and $v \in V$. Since Φ is $\Bbbk$-linear and Laurent polynomials are dense in Laurent series, this reduces to showing that $\Phi(\theta^i) = \varphi(\theta^i)\Phi$, which can be proved by induction as long as you can show that $\Phi(\theta) = \left(\frac{\theta}{1+\theta}\right)\Phi$. This last equation is equivalent to $(\eta+1)\Phi = \Phi\eta$, which we now prove. Using the fact that $[\nabla, z] = 1$ and the definitions given in (3.1) we compute

$$(\eta+1)\Phi = -z\nabla z + z = -z(z\nabla + 1) + z = z(-z\nabla) = \Phi\eta.$$

$\square$

Lemma 3.3.3. *The definition for θ, as defined in* (3.1)*, extends to an action of* $\Bbbk((\theta))$ *on V.*

Proof. Since all indecomposable components of ∇ have positive slope, ∇ (and $z\nabla$) will be invertible and thus θ is well-defined. An action of $\Bbbk[\theta^{-1}] = \Bbbk[-z\nabla]$ on V is trivially defined. If $z\nabla : V \to V$ is nicely expanding, then by Proposition 3.1.5 we will also have an action of $\Bbbk[[\theta]]$ on V. This will give a well defined action of $\Bbbk((\theta))$ on V. Thus all we need to prove is that $z\nabla : V \to V$ is nicely expanding.

We must show that $\theta = (z\nabla)^{-1} : V \to V$ is continuous, open, linearly compact and contracting. Thanks to the canonical form we can assume without loss of generality that ∇ is indecomposable. Thus, in canonical form, $z\nabla$ will be

$$z\frac{d}{dz} + \begin{bmatrix} f & & \\ 1 & \ddots & \\ & \ddots & \ddots \end{bmatrix}$$

with $f \in \Bbbk[z^{-1/r}]$ and $\mathrm{ord}(f) = -m/r < 0$. Let $\{e_i\}$ be the canonical basis. Since lattices are linearly compact open subspaces, to prove that $(z\nabla)^{-1}$ is open, continuous and linearly compact it suffices to show that $(z\nabla)$ and $(z\nabla)^{-1}$ map a lattice of the form $L_k = \bigoplus (z^{1/r})^k A_r e_i$ to a lattice of the same form (note that here we are using $A_r = \Bbbk[[z^{1/r}]]$). We see that $z\nabla(L_k) = \bigoplus (z^{1/r})^{k-m} A_r e_i = L_{k-m}$ and $(z\nabla)^{-1}(L_k) = \bigoplus (z^{1/r})^{k+m} A_r e_i = L_{k+m}$, so $(z\nabla)^{-1}$ is open, continuous and linearly compact.

To show that $(z\nabla)^{-1}$ is contracting, by Lemma 3.2.1 we only need to show that $\mathrm{Ord}((z\nabla)^{-1}) > 0$. By Corollary 1.4.15, (1), then, it suffices to show that we have $\mathrm{ord}(f) < 0$ for the indecomposable $(V, \nabla) = E_f \otimes J_m$. This condition is fulfilled by assumption, since all indecomposable components have slope greater than zero. $\square$

3.3.2. Definition of $\mathcal{M}^{(x,\infty)}$.

Definition 3.3.4. Let $E = (V, \nabla) \in \mathcal{C}_x$ such that ∇ has no horizontal sections. Consider on V the $\Bbbk$-linear operators

$$\theta := -(z\nabla)^{-1} : V \to V \text{ and } \Phi := z : V \to V \tag{3.2}$$

Then θ extends to an action of $\Bbbk((\theta))$ on V, $\dim_{\Bbbk((\theta))} V < \infty$, and Φ is an invertible difference operator. We define *the local Mellin transform from x to infinity of E* to be the object

$$\mathcal{M}^{(x,\infty)}(E) := (V_\theta, \Phi) \in \mathcal{N}.$$

Remark. Note that since $E \in \mathcal{C}_x$, we are thinking of K as $\Bbbk((z_x))$. This emphasizes that we are localizing at a point $x \neq 0$ with local coordinate $z_x = z - x$.

Claim 3.3.5. $\mathcal{M}^{(x,\infty)}$ *is well-defined.*

Proof. To prove the claim we must show the following:

(i) θ extends to an action of $\Bbbk((\theta))$ on V.

(ii) V_θ is finite dimensional.

(iii) Φ is an invertible difference operator on V_θ.

We prove (i) with Lemma 3.3.6 below. The proofs of (ii) and (iii) are identical to those found in the proof of Claim 3.3.2. □

Lemma 3.3.6. *The definition for* θ*, as given in* (3.2) *extends to an action of* $\Bbbk((\theta))$ *on* V*.*

Proof. As in the proof of Claim 3.3.2, all we need is that $z\nabla : V \to V$ is nicely expanding. Since ∇ has no horizontal sections, ∇ (and thus $z\nabla$) will be invertible. First we show that $(z\nabla)^{-1} : V \to V$ is continuous, open, and linearly compact. We can assume without loss of generality that, in canonical form, ∇ is a single indecomposable block. Thus $z\nabla$ will have the form

$$z\frac{d}{dz_x} + \frac{z}{z_x}\begin{bmatrix} f & & \\ 1 & \ddots & \\ & \ddots & \ddots \end{bmatrix}$$

with $f \in \Bbbk[z_x^{-1/r}]$ and $\operatorname{ord}(f) = (-s/r) \leq 0$. It suffices to show that $(z\nabla)$ and $(z\nabla)^{-1}$ map a lattice of the form $L_k = \bigoplus (z_x^{1/r})^k A e_i$ to a lattice of the same form (note that here we are using $A = \Bbbk[[z_x^{1/r}]]$). It is helpful to note that the leading term of an operator is often the only important term for theoretical calculation. Thus one can think of z as

z_x+x, and reduce to considering $z\nabla$ as merely $x\nabla$ to conclude that $z\nabla(L_k) = L_{k-s-r}$ and $(z\nabla)^{-1}(L_k) = L_{k+s+r}$. It follows that $(z\nabla)^{-1}$ is open, continuous and linearly compact.

To show that $(z\nabla)^{-1}$ is contracting, by Lemma 3.2.1 we need $\mathrm{Ord}((z\nabla)) > 0$. By Corollary 1.4.15, (2) it suffices to show that we have $\mathrm{ord}(f) < 1$ for the indecomposable $(V, \nabla) = E_f \otimes J_m$. This condition is trivially fulfilled since the slope of a connection is nonnegative. $\square$

3.3.3. Definition of $\mathcal{M}^{(\infty,\infty)}$. Note that here we are thinking of K as $\Bbbk((\zeta))$, since we are localizing at the point at infinity $\zeta = \frac{1}{z}$.

Definition 3.3.7. Let $E = (V, \nabla) \in \mathcal{C}_{\infty}^{>0}$, thus all irreducible components of ∇ have slope greater than zero. Consider on V the $\Bbbk$-linear operators

$$\theta := -(z\nabla)^{-1} : V \to V \text{ and } \Phi := z : V \to V \tag{3.3}$$

Then θ extends to an action of $\Bbbk((\theta))$ on V, $\dim_{\Bbbk((\theta))} V < \infty$, and Φ is an invertible difference operator. We define *the local Mellin transform from infinity to infinity of E* to be the object

$$\mathcal{M}^{(\infty,\infty)}(E) := (V_\theta, \Phi) \in \mathcal{N}.$$

Claim 3.3.8. *$\mathcal{M}^{(\infty,\infty)}$ is well-defined.*

Proof. The proof of Claim 3.3.8 is almost identical to the proof of Claim 3.3.2 (with one caveat mentioned below) and is thus omitted. The 'caveat' was described previously in the remark before Proposition 1.4.11, but we repeat the important parts here. In particular we note that at infinity, due to the change of variable from z to ζ, we write the canonical form for an indecomposable ∇_z as

$$-\zeta^2\nabla_\zeta = -\zeta^2\frac{d}{d\zeta} + \zeta \begin{bmatrix} f & & \\ 1 & \ddots & \\ & \ddots & \ddots \end{bmatrix}$$

with $f \in \Bbbk[\zeta^{-1/r}]$. Thus $z\nabla_z = \zeta\nabla_\zeta$ will be

$$-\zeta\frac{d}{d\zeta} + \begin{bmatrix} f & & \\ 1 & \ddots & \\ & \ddots & \ddots \end{bmatrix}.$$

$\square$

Remark. The local Mellin transforms above give functors to apply to all connections except for certain connections with regular singularity. More precisely, the only invertible connections for which $\mathcal{M}^{(0,\infty)}$, $\mathcal{M}^{(x,\infty)}$, and $\mathcal{M}^{(\infty,\infty)}$ cannot be applied are those connections in $\mathcal{C}_0$ and $\mathcal{C}_\infty$ with slope zero. It turns out that these connections with regular singularity will map to difference operators with singularity at a point $y \neq \infty$. This case is sufficiently small and different from the situation described above that we do not discuss it in this dissertation.

3.4. Definition of local inverse Mellin transforms

3.4.1. Definition of $\mathcal{M}^{-(0,\infty)}$.

Definition 3.4.1. Let $D = (V, \Phi) \in \mathcal{N}^{>0}$. Thus Φ is invertible and the irreducible components of Φ have order greater than zero. Consider on V the $\Bbbk$-linear operators

$$z := \Phi : V \to V \text{ and } \nabla := -(\theta\Phi)^{-1} : V \to V \tag{3.4}$$

Then z extends to an action of $\Bbbk((z))$ on V, $\dim_{\Bbbk((z))} V < \infty$, and ∇ is a connection. We write V_z for V to denote that we are considering V as a $\Bbbk((z))$-vector space. We define *the local inverse Mellin transform from zero to infinity of D* to be the object

$$\mathcal{M}^{-(0,\infty)}(D) := (V_z, \nabla) \in \mathcal{C}_0.$$

Claim 3.4.2. $\mathcal{M}^{-(0,\infty)}$ *is well-defined.*

Proof. We must show the following:

(i) z extends to an action of $\mathbb{k}((z))$ on V.

(ii) V_z is finite dimensional.

(iii) ∇ is a connection on V_z.

We prove (i) with Lemma 3.4.3 below. In the proof of Lemma 3.4.3 we show that V_z is of Tate type. Lemma 3.2.2 then implies that V_z is finite-dimensional, proving (ii). To prove (iii) we must show that $[\nabla, f] = f'$ for all $f \in \mathbb{k}((z))$. We first show that it suffices to prove that $[\nabla, z] = 1$. Since ∇ is $\mathbb{k}$-linear and Laurent polynomials are dense in Laurent series, to show that $[\nabla, f] = f'$ we merely need to show that $[\nabla, z^n] = nz^{n-1}$ for all $n \in \mathbb{Z}$. Assuming that $[\nabla, z] = 1$, though, we can prove that $[\nabla, z^n] = nz^{n-1}$ for all $n \in \mathbb{Z}$ with the following steps: an induction argument proves it true for $n > 1$. Then using the fact that $0 = [\nabla, 1] = [\nabla, zz^{-1}]$ one can derive $[\nabla, z^{-1}] = -z^{-2}$. An induction argument then proves $[\nabla, z^n] = nz^{n-1}$ for the negative integers. All that remains is to show that $[\nabla, z] = 1$. We note that for $\eta = \theta^{-1}$ we have $\Phi^{-1}(\eta)\Phi = (\eta - 1)$. Then we can use the definitions of (3.4) to compute

$$\nabla z - z\nabla = -(\theta\Phi)^{-1}\Phi + \Phi(\theta\Phi)^{-1} = -\Phi^{-1}\eta\Phi + \eta = 1.$$

□

Lemma 3.4.3. *The definition of z, as given in* (3.4)*, extends to an action of $\mathbb{k}((z))$ on V.*

Proof. Φ is invertible, so an action of $\mathbb{k}[z^{-1}]$ is defined. We prove that Φ is nicely contracting and then invoke Proposition 3.1.5 to show that an action of $\mathbb{k}[[z]]$ is well-defined.

To apply Proposition 3.1.5, we need $z = \Phi$ to be continuous, open, linearly compact, and contracting. First we show that Φ is open, continuous and linearly compact. We can assume that Φ is indecomposable, so in canonical form $(V, \Phi) = D_g \otimes T_m$ for some $g \in K_r$ with $\mathrm{ord}(g) = s/r$. Let $\{e_i\}$ be the canonical basis. As in previous proofs, it suffices to show that Φ and Φ^{-1} map a lattice of the form $L_k = \bigoplus(\theta^{1/r})^k Ae_i$ to a lattice of the same form (note that here we are using $A = \Bbbk[[\theta^{1/r}]]$). Calculation using the canonical form shows that $\Phi(L_k) = L_{k+s}$ and $\Phi^{-1}(L_k) = L_{k-s}$, so Φ is open, continuous and linearly compact.

To show that an indecomposable Φ is contracting, by Lemma 3.2.1 we need to show that $\mathrm{Ord}(\Phi) > 0$. By Corollary 1.4.15, (3), we simply need that for $(V, \Phi) = D_g \otimes T_m$ we have $\mathrm{ord}(g) > 0$. This follows from the assumption that all irreducible components of Φ have order greater than zero.

$\square$

3.4.2. Definition of $\mathcal{M}^{-(x,\infty)}$.

Definition 3.4.4. Let $D = (V, \Phi) \in \mathcal{N}^{=0}$ such that all irreducible components of Φ have order zero with the same leading coefficient $x \neq 0$, and $\Phi - x$ is invertible. Consider on V the $\Bbbk$-linear operators

$$z := \Phi : V \to V \text{ and } \nabla := -(\theta\Phi)^{-1} : V \to V. \tag{3.5}$$

Then the action of $z - x = z_x$ is clearly defined, z_x extends to an action of $\Bbbk((z_x))$ on V, $\dim_{\Bbbk((z_x))} V < \infty$, and ∇ is a connection. We write V_{z_x} for V to denote that we are considering V as a $\Bbbk((z_x))$-vector space. We define *the local inverse Mellin transform from x to infinity of D* to be the object

$$\mathcal{M}^{-(x,\infty)}(D) := (V_{z_x}, \nabla) \in \mathcal{C}_x.$$

Claim 3.4.5. *$\mathcal{M}^{-(x,\infty)}$ is well-defined.*

Proof. To prove the claim we must show the following:

(i) z_x extends to an action of $\Bbbk((z_x))$ on V.
(ii) V_{z_x} is finite dimensional.
(iii) ∇ is a connection on V_{z_x}.

We prove (i) with Lemma 3.4.6 below. In the proof of Lemma 3.4.6 we show that V_{z_x} is of Tate type. Lemma 3.2.2 then implies that V_{z_x} is finite-dimensional, proving (ii).

To prove (iii) we simply note that $[\nabla, z_x] = [\nabla, z]$ and then refer to the proof of Claim 3.4.2,(iii). $\square$

Lemma 3.4.6. *The definition for z_x, as given in* (3.5)*, extends to an action of* $\Bbbk((z_x))$ *on V.*

Proof. $z_x = \Phi - x$ is invertible, so an action of $\Bbbk[z_x^{-1}]$ is defined. To show that the action of $\Bbbk[[z_x]]$ is well-defined, we prove that z_x is nicely contracting and then invoke Proposition 3.1.5.

We first show that z_x is continuous, open, and linearly compact. We can assume that Φ is indecomposable, so according to Theorem 1.2.3, in canonical form $z_x = \Phi - x$ will be

$$\begin{bmatrix} g & & \\ \theta^{\lambda} & \ddots & \\ & \ddots & \ddots \end{bmatrix} \varphi$$

where $g \in \Bbbk[[\theta^{1/r}]]$ and $\operatorname{ord}(g) = s/r > 0$. Let $\{e_i\}$ be the canonical basis. As in previous proofs, it suffices to show that z_x and z_x^{-1} map a lattice of the form $L_k = \bigoplus(\theta^{1/r})^k A e_i$ to a lattice of the same form (note that here we are using $A = \Bbbk[[\theta^{1/r}]]$). Calculation using the canonical form shows that $\Phi(L_k) = L_{k+s}$ and $\Phi^{-1}(L_k) = L_{k-s}$, so z_x is open, continuous and linearly compact.

To see that z_x is contracting, we note that although $\operatorname{ord}(g) = 0$, we have $\operatorname{ord}(g-x) > 0$. Thus by Corollary 1.4.15, (3) we see that z_x is an indecomposable difference operator with $\operatorname{Ord}(z_x) > 0$. Lemma 3.2.1 then confirms that z_x is contracting.

$\square$

3.4.3. Definition of $\mathcal{M}^{-(\infty,\infty)}$.

Definition 3.4.7. Let $D = (V, \Phi) \in \mathcal{N}^{<0}$. Thus Φ is invertible and the irreducible components of Φ have order less than zero. Consider on V the $\Bbbk$-linear operators

$$z := \Phi : V \to V \text{ and } \nabla := -(\theta\Phi)^{-1} : V \to V. \tag{3.6}$$

Then $\zeta = z^{-1}$ extends to an action of $\Bbbk((\zeta))$ on V and $\dim_{\Bbbk((\zeta))} V < \infty$. We write V_ζ for V to denote that we are considering V as a $\Bbbk((\zeta))$-vector space. We define *the local inverse Mellin transform from infinity to infinity of* D to be the object

$$\mathcal{M}^{-(\infty,\infty)}(D) := (V_\zeta, \nabla) \in \mathcal{C}_\infty.$$

Claim 3.4.8. $\mathcal{M}^{-(\infty,\infty)}$ *is well-defined.*

Proof. The fact that all irreducible components of Φ have order less than zero implies that $\mathrm{Ord}(\Phi) < 0$. This in turn implies that $\mathrm{Ord}(\Phi^{-1}) = \mathrm{Ord}(\zeta) > 0$, and the remainder of the proof is identical to the proof of Claim 3.4.2. □

3.5. Equivalence of categories

Assuming that composition of the functors is defined, by inspection one can see that $\mathcal{M}^{(0,\infty)}$ and $\mathcal{M}^{-(0,\infty)}$ are inverse functors (and the same holds for the pairs $\mathcal{M}^{(x,\infty)}$ and $\mathcal{M}^{-(x,\infty)}$ as well as $\mathcal{M}^{(\infty,\infty)}$ and $\mathcal{M}^{-(\infty,\infty)}$). Thus in order to show that the local Mellin transforms induce certain equivalences of categories, all we need is to confirm that the functors map into the appropriate subcategories. We first prove an important property of normed vector spaces which coincides with properties of Tate vector spaces. This will be useful in demonstrating the equivalence of categories.

3.5.1. Normed vector spaces. Our first goal is to prove the following lemma, which will greatly simplify the relationship between the norm of an operator and its local Mellin transform. First we give some definitions related to infinite-dimensional vector spaces over $\Bbbk$.

Definition 3.5.1. Let V be an infinite-dimensional vector space over $\Bbbk$. A *norm* on V is a real-valued function $||\bullet||$ such that the following hold:

(1) $||v|| > 0$ for $v \in V - \{0\}$, $||0|| = 0$.
(2) $||v + w|| \leq \max(||v||, ||w||)$ for all $v, w \in V$.
(3) $||c \cdot v|| = ||v||$ for $c \in \Bbbk$ and $v \in V$.

Definition 3.5.2. An infinite-dimensional vector space V over $\Bbbk$ is *locally linearly compact* if for any $r_1 > r_2 > 0$, $r_i \in \mathbb{R}$, the ball of radius r_2 has finite codimension in the ball of radius r_1.

Proposition 3.5.3. *Let V be an infinite-dimensional vector space over $\Bbbk$, equipped with a norm $||\bullet||$ such that V is complete in the induced topology. Let $0 < \epsilon < 1$ and $Y : V \to V$ be an invertible $\Bbbk$-linear operator such that $||Y|| = \epsilon^\alpha < 1$ and $||Y^{-1}|| = \epsilon^{-\alpha}$. Define $\hat{\epsilon} := \epsilon^\alpha$. Then*

(1) V has a unique structure of a $K = \Bbbk((y))$-vector space such that y acts as Y and the norm $||\bullet||$ agrees with the valuation on K where $|f| = \hat{\epsilon}^{ord(f)}$ for $f \in K$.

(2) V is finite-dimensional over K if and only if V is locally linearly compact.

Proof. (1) The action of K is defined by $(\sum_j c_i y^i)v = \sum c_i(Y^i v)$. This action is clearly defined for Laurent polynomials $\sum_{i=k} c_i y^i$. The definition extends to series $\sum_{i=k}^{\infty} c_i y^i$ because $||Y^n|| = (\epsilon^\alpha)^n \to 0$ as n goes to infinity, which follows from Corollary 1.4.8 and the fact that $||Y|| = ||Y||_{inf} = \epsilon < 1$. The fact that $||\bullet||$ agrees with the valuation on K follows by construction: because Y is a similitude we have $||Y^n(v)|| = \epsilon^{\alpha n}||v|| = \hat{\epsilon}^n||v|| = |y^n|||v||$.

For uniqueness, suppose there is a $\Bbbk((y'))$-vector space structure such that y' acts as Y and $||\bullet||$ coincides with the valuation on $\Bbbk((y'))$. It is clear that the two structures coincide for Laurent polynomials, so we only need to show the same for infinite series. Specifically, we wish to show the following for any $v \in V$: If $\left(\sum_{i=k}^{\infty} c_i y^i\right) v = \ell$ and $\left(\sum_{i=k}^{\infty} c_i (y')^i\right) v = \ell'$, then $\ell = \ell'$. Let $\left(\sum_{i=n}^{\infty} c_i y^i\right) v = w_n$ and $\left(\sum_{i=n}^{\infty} c_i (y')^i\right) v = w'_n$. Since the norm agrees with the valuation, we have

$$\lim_{n\to\infty} ||w_n|| = \lim_{n\to\infty} ||w'_n|| = \lim_{n\to\infty} |y^n| = 0,$$

So by completeness, both $\{w_n\}$ and $\{w'_n\}$ converge to zero. We now write $\ell = w_n + \left(\sum_{i=k}^{n-1} c_i y^i\right) v$ and $\ell' = w'_n + \left(\sum_{i=k}^{n-1} c_i (y')^i\right) v$. Since the action coincides on Laurent polynomials, it follows that $\ell - \ell' = w_n - w'_n$ must equal zero.

(2) We prove the forward direction as follows. Suppose V is finite dimensional and $V \simeq K^n$. For $v \in V$ we write $v = \sum v_i e_i$ and we use the norm given in Example 1.4.3 for K^n. Let $r_1 > r_2 > 0$ be constants, and write B_{r_j} to denote the ball of radius r_j. Then for $j = 1, 2$, we have $B_{r_j} = \left\{v \in V \middle| ||v|| = \max\{|v_i|\} < r_j\right\} = \left\{v \in V \middle| \min\{\text{ord}(v_i)\} \geq m_j\}\right\}$ for some $m_j \in \mathbb{Z}$ with $m_1 < m_2$. It follows that the codimension of B_{r_2} in B_{r_1} will be $n(m_2 - m_1) < \infty$.

For the reverse direction, suppose V is infinite-dimensional over K. Then there exists a infinite basis $\{e_i\}$ such that $e_i \notin \text{Span}\langle e_1, \ldots, e_{i-1}\rangle$ for all i. Consider the balls $B = \prod \mathbb{k}[[y]]e_i$ and $B' = \prod \mathbb{k}[[y]]ye_i$. Then B' has infinite codimension in B, so V is not locally linearly compact.

□

Remark. If V is a Tate vector space then the unique structure of Proposition 3.5.3, (1) coincides with that of Proposition 3.1.5.

Corollary 3.5.4. *Let V be a $\mathbb{k}((y))$-vector space, $Z : V \to V$ a similitude, and $||Z|| = ||Z||_{inf} = \epsilon^\alpha < 1$. Then V can be considered as a $\mathbb{k}((Z))$-vector space (in the spirit of Proposition 3.5.3) and for any similitude $A : V \to V$ we have $||A|| = ||A||_Z$. In particular, A will be a similitude when V is viewed as either a $\mathbb{k}((y))$- or $\mathbb{k}((Z))$-vector space.*

PROOF. This follows directly from Proposition 3.5.3 because we take the valuation on $\mathbb{k}((Z))$ by using $\hat{\epsilon} = \epsilon^\alpha$ instead of ϵ. □

3.5.2. Lemmas.

Lemma 3.5.5. *The local Mellin transforms and their inverses map indecomposable objects to indecomposable objects.*

PROOF. We give the proof for $\mathcal{M}^{(0,\infty)}$, the proofs for the others are identical. Suppose that $\mathcal{M}^{(0,\infty)}(V,\nabla) = (V_\theta, \Phi)$ and $V_\theta = W_1 \oplus W_2$ for nonzero subspaces W_i. Then by the canonical form we have that $\Phi(W_i) \subset W_i$ and $\Phi^{-1}(W_i) \subset W_i$. Since V_θ is a $\Bbbk((\theta))$-vector space we also trivially have that $\theta(W_i) \subset W_i$. By definition of $\mathcal{M}^{(0,\infty)}$, this means that $z(W_i) \subset W_i$ and $-z\nabla(W_i) \subset W_i$. In particular, it follows that $\nabla(W_i) \subset W_i$, so V will be decomposable. This implies that if the local Mellin transform of an object is decomposable, the original object is decomposable as well, and the result follows. $\square$

Lemma 3.5.6. *Let $E = (V,\nabla) \in \mathcal{C}_0^{>0}$, θ, and Φ be as given in Definition 3.3.1. Then $\mathcal{M}^{(0,\infty)}(E) \in \mathcal{N}^{>0}$.*

PROOF. Due to the canonical decomposition it suffices to prove the lemma when E is indecomposable. Then ∇ and z are similitudes, so by Corollary 3.5.4, θ and Φ are also similitudes. By Lemma 3.5.5, $\mathcal{M}^{(0,\infty)}(E)$ is indecomposable, so to prove Lemma 3.5.6 it suffices to show that $||\Phi||_\theta < 1$. By Corollary 3.5.4 we have that $||A||_z = ||A||_\theta$ for any similitude A, and it follows that

$$||\Phi||_\theta = ||z||_z = (\epsilon)^1 < 1.$$

$\square$

Lemma 3.5.7. *Let $D = (V,\Phi) \in \mathcal{N}^{>0}$, z, and ∇ be as given in Definition 3.4.1. Then $\mathcal{M}^{-(0,\infty)}(D) \in \mathcal{C}_0^{>0}$.*

PROOF. Due to the canonical decomposition it suffices to prove the lemma when D is indecomposable. Then Φ and θ are similitudes, so by Corollary 3.5.4, z and ∇ are also similitudes. By Lemma 3.5.5, $\mathcal{M}^{-(0,\infty)}(D)$ is indecomposable. Thus to prove Lemma 3.5.7 it suffices to show that $||z\nabla|| > 1$. By Corollary 3.5.4 we have that $||A||_z = ||A||_\theta$ for any similitude A, and it follows that

$$||z\nabla||_z = ||\Phi(-\theta\Phi)^{-1}||_\theta = ||\theta^{-1}||_\theta = (\epsilon)^{-1} > 1.$$

$\square$

Lemma 3.5.8. *Let $E = (V, \nabla) \in \mathcal{C}_x$, θ, and Φ be as given in Definition 3.3.4. Then $\mathcal{M}^{(x,\infty)}(E) \in \mathcal{N}^{=0}$.*

Proof. The general reasoning is identical to the proof of Lemma 3.5.6, so we only point out the differences in the calculations. We need to show that $||\Phi||_\theta = 1$. Since the local coordinate is z_x, multiplication by z is an isometry and we have

$$1 = ||z||_{z_x} = ||\Phi||_\theta.$$

$\square$

Lemma 3.5.9. *Let $E = (V, \nabla) \in \mathcal{C}_\infty^{>0}$, θ, and Φ be as given in Definition 3.3.7. Then $\mathcal{M}^{(\infty,\infty)}(E) \in \mathcal{N}^{<0}$.*

Proof. The general reasoning is identical to the proof of Lemma 3.5.6, so we only point out the differences in the calculations. We need to show that $||\Phi||_\theta > 1$, which follows from the calculation

$$||\Phi||_\theta = ||z||_\zeta = ||\zeta^{-1}||_\zeta = \epsilon^{-1} > 1.$$

$\square$

Lemma 3.5.10. *Let $D = (V, \Phi) \in \mathcal{N}^{<0}$, z, and ∇ be as given in Definition 3.4.7. Then $\mathcal{M}^{-(\infty,\infty)}(D) \in \mathcal{C}_\infty^{>0}$.*

Proof. The general reasoning is identical to the proof of Lemma 3.5.7, so we only point out the differences in the calculations. It suffices to show that $||z\nabla||_\zeta > 1$, which follows from

$$||z\nabla||_\zeta = ||\Phi(-\theta\Phi)^{-1}||_\theta = ||-\theta^{-1}||_\theta = \epsilon^{-1} > 1.$$

$\square$

3.5.3. Proofs for equivalence of categories.

Theorem 3.5.11. *The local Mellin transform $\mathcal{M}^{(0,\infty)}$ induces an equivalence of categories between $\mathcal{C}_0^{>0}$ and $\mathcal{N}^{>0}$.*

Proof. This follows from Lemmas 3.5.6 and 3.5.7, as well as the fact (stated above) that $\mathcal{M}^{(0,\infty)}$ and $\mathcal{M}^{-(0,\infty)}$ are inverse functors. □

Theorem 3.5.12. *The local Mellin transform $\mathcal{M}^{(x,\infty)}$ induces an equivalence of categories between the subcategory of $\mathcal{C}_x$ of connections with no horizontal sections and $\mathcal{N}^{=0}$.*

Proof. This follows from Lemma 3.5.8 and the fact that ∇ (as defined in Definition 3.4.4) is invertible and thus has no horizontal sections, as well as the fact that $\mathcal{M}^{(x,\infty)}$ and $\mathcal{M}^{-(x,\infty)}$ are inverse functors. □

Theorem 3.5.13. *The local Mellin transform $\mathcal{M}^{(\infty,\infty)}$ induces an equivalence of categories between $\mathcal{C}_\infty^{>0}$ and $\mathcal{N}^{<0}$.*

Proof. This follows from Lemmas 3.5.9 and 3.5.10, as well as the fact that $\mathcal{M}^{(\infty,\infty)}$ and $\mathcal{M}^{-(\infty,\infty)}$ are inverse functors. □

CHAPTER 4

Explicit formulas for local Mellin transforms

In this chapter we give precise statements and proofs of explicit formulas for calculating the local Mellin transforms and their inverses. The results and proofs found in this chapter are analogous to those given in Chapter 2 regarding the local formal Fourier transforms.

In section 4.1 we state explicit formulas for calculating the local Mellin transforms. In section 4.2 we give an explicit formula for the local inverse Mellin transform $\mathcal{M}^{-(0,\infty)}$ and explain how to derive formulas for the other local inverse Mellin transforms. Section 4.3 is devoted to proving the formulas given in section 4.1.

4.1. Statement of theorems for local Mellin transforms

4.1.1. Calculation of $\mathcal{M}^{(0,\infty)}$.

Theorem 4.1.1. *Let s and r be positive integers, $a \in \Bbbk - \{0\}$, and $f \in R_r^\circ(z)$ with $f = az^{-s/r} + \underline{o}(z^{-s/r})$. Then*

$$\mathcal{M}^{(0,\infty)}(E_f) \simeq D_g,$$

where $g \in S_s^\circ(\theta)$ is determined by the following system of equations:

$$f = -\theta^{-1} \tag{4.1}$$

$$g = z - (-a)^{r/s}\left(\frac{r+s}{2s}\right)\theta^{1+(r/s)} \tag{4.2}$$

Remark. We determine g using (4.1) and (4.2) as follows. One can think of (4.1) as an implicit definition for the variable z. Thus we first use (4.1) to give an explicit expression for z in terms of $\theta^{1/s}$. We then substitute this explicit expression into (4.2) to get an expression for $g(\theta)$ in terms of $\theta^{1/s}$.

When we use (4.1) to write an expression for z in terms of $\theta^{1/s}$, the expression is not unique since we must make a choice of a root of unity. More concretely, let η be a primitive s^{th} root of unity. Then replacing $\theta^{1/s}$ with $\eta\theta^{1/s}$ in our explicit equation for z will yield another possible expression for z. This choice will not affect the overall result, however, since all such possible expressions will lie in the same Galois orbit. Thus by Proposition 1.2.5,(1), any choice of root of unity will correspond to the same difference operator.

Corollary 4.1.2. *Let E be an object in $\mathcal{C}_0^{>0}$. By Proposition 1.1.4,* (3)*, let E have decomposition $E \simeq \bigoplus_i \Big(E_{f_i} \otimes J_{m_i}\Big)$ where all E_{f_i} have positive slope. Then*

$$\mathcal{M}^{(0,\infty)}(E) \simeq \bigoplus_i \Big(D_{g_i} \otimes T_{m_i}\Big)$$

where $D_{g_i} = \mathcal{M}^{(0,\infty)}(E_{f_i})$ for all i.

Sketch of Proof. The equivalence of categories given in Theorem 3.5.11 implies that

$$\mathcal{M}^{(0,\infty)}\left[\bigoplus_i \Big(E_{f_i} \otimes J_{m_i}\Big)\right] \simeq \bigoplus_i \mathcal{M}^{(0,\infty)}\Big(E_{f_i} \otimes J_{m_i}\Big).$$

The equivalence also implies that $\mathcal{M}^{(0,\infty)}$ will map the indecomposable object $E_f \otimes J_m$ (as the unique indecomposable in $\mathcal{C}_0$ formed by m successive extensions of E_f) to an indecomposable object $D_g \otimes T_m$ (as the unique indecomposable in $\mathcal{N}$ formed by m successive extensions of D_g). It follows that we only need to know how $\mathcal{M}^{(0,\infty)}$ acts on E_f, which is given by Theorem 4.1.1. $\square$

Remark. Analogous corollaries hold for the calculation of the other local Mellin transforms, however we do not state them explicitly.

4.1.2. Calculation of $\mathcal{M}^{(x,\infty)}$.

Theorem 4.1.3. *Let s be a nonnegative integer, r a positive integer, and $a \in \Bbbk - \{0\}$. Let $f \in R_r^\circ(z_x)$ with $f = a z_x^{-s/r} + \underline{o}(z_x^{-s/r})$. Then*

$$\mathcal{M}^{(x,\infty)}(E_f) \simeq D_g,$$

where $g \in S_{r+s}^\circ(\theta)$ is determined by the following system of equations:

$$f = -\left(\frac{z_x}{z}\right)\theta^{-1} \tag{4.3}$$

$$g = z + \left(\frac{xs}{2(s+r)}\right)\theta \tag{4.4}$$

Remark. We determine g using (4.3) and (4.4) as follows. First, using (4.3) we explicitly express z in terms of $\theta^{1/(r+s)}$. We then substitute this explicit expression for z into (4.4) to get an expression for $g(\theta)$ in terms of $\theta^{1/(r+s)}$.

4.1.3. Calculation of $\mathcal{M}^{(\infty,\infty)}$.

Theorem 4.1.4. *Let s and r be positive integers and $a \in \Bbbk - \{0\}$. Then for $f \in R_r^\circ(\zeta)$ with $f = a\zeta^{-s/r} + \underline{o}(\zeta^{-s/r})$ we have*

$$\mathcal{M}^{(\infty,\infty)}(E_f) \simeq D_g,$$

where $g \in S_s^\circ(\theta)$ is determined by the following system of equations:

$$f = -\theta^{-1} \tag{4.5}$$

$$g = z - (-a)^{r/s}\left(\frac{r+s}{2s}\right)\theta^{1-(r/s)} \tag{4.6}$$

Remark. We determine g using (4.5) and (4.6) as follows. First, using (4.5) we express z in terms of $\theta^{1/s}$. We then substitute this expression into (4.6) and solve to get an expression for $g(\theta)$ in terms of $\theta^{1/s}$.

4.2. Statement of theorems for local inverse Mellin transforms

In Chapter 3 we explained that $\mathcal{M}^{-(0,\infty)}$, $\mathcal{M}^{-(x,\infty)}$, and $\mathcal{M}^{-(\infty,\infty)}$ are inverse functors for (respectively) $\mathcal{M}^{(0,\infty)}$, $\mathcal{M}^{(x,\infty)}$, and $\mathcal{M}^{(\infty,\infty)}$. It follows that explicit formulas for the local inverse Mellin transforms can be found merely by "inverting" the expressions found in Theorems 4.1.1, 4.1.3, and 4.1.4. We give an example below of what this would look like for $\mathcal{M}^{-(0,\infty)}$, the other local inverse Mellin transforms are similar. The proofs are omitted.

Theorem 4.2.1. *Let p and q be positive integers and $g \in S_q^{\circ}(\theta)$ with $g = a\theta^{p/q} + \underline{o}(\theta^{p/q})$, $a \neq 0$. Then*

$$\mathcal{M}^{-(0,\infty)}(D_g) \simeq E_f,$$

where $f \in R_p^{\circ}(z)$ is determined by the following system of equations:

$$g + a\left(\frac{p+q}{2q}\right)\theta^{1+(p/q)} = z \tag{4.7}$$

$$f = -\theta^{-1} \tag{4.8}$$

Remark. We determine f using (4.7) and (4.8) as follows. First, using (4.7) we explicitly express θ in terms of $z^{1/p}$. We then substitute this explicit expression for θ into (4.8) and solve to get an expression for $f(z)$ in terms of $z^{1/p}$.

4.3. Proof of theorems

Outline. We begin with a brief outline of the proof for Theorem 4.1.1. Starting with Definition 3.4.1 of $\mathcal{M}^{(0,\infty)}$, we set $\theta = -(z\nabla)^{-1}$ and $\Phi = z$. For irreducible objects E_f and D_g we have $\nabla = \frac{d}{dz} + z^{-1}f$ and $\Phi = g\varphi$, and our goal is to use the given value of f to find the expression for g. Since $z = z(1) = \Phi(1) = g\varphi(1) = g$, this amounts to finding an expression for the operator z in terms of the operator θ. The equation $\theta = -(z\nabla)^{-1}$ gives an expression for θ in terms of z, and we use the Operator-root

Lemma (1.5.3) to write an explicit expression for the operator z in terms of θ. The calculation primarily involves finding particular fractional powers of f, but we must also keep track of the interplay between the linear and differential parts of ∇ during the calculation; this interplay accounts for the subtraction of the term $(-a)^{r/s}\left(\frac{r+s}{2s}\right)\theta^{1+\frac{r}{s}}$ from our expression for g.

The proofs for Theorems 4.1.3 and 4.1.4 are similar and thus outlines for their proofs are omitted. The only change of note is that in the proof of Theorem 4.1.3 we must also prove a separate case for when our connection is regular singular (i.e. when $\mathrm{ord}(f) = 0$).

Remark. We give a brief explanation regarding the origin of the system of equations found in Theorem 4.1.1. In the same fashion as described in the remark of subsection 2.3.1, consider the equations in (3.1). As before, let $\nabla = z^{-1}f$ (i.e. as normally defined but without the differential part) and $\Phi = g$ (i.e. as normally defined but without the shift operator φ). Then the equations $f = -\theta^{-1}$ and $g = z$ fall out easily. The reason the extra term shows up in (4.2) is due to the interaction of the linear and differential parts of ∇, as described above in the outline.

4.3.1. Proof of Theorem 4.1.1.

PROOF. Given $\theta = -(z\nabla)^{-1}$ and $\nabla = \frac{d}{dz} + z^{-1}f$, we find that

$$-\theta = \left(z\frac{d}{dz} + f\right)^{-1}. \tag{4.9}$$

We wish to express the operator z in terms of the operator θ.

Consider the equation

$$-\theta = f^{-1}, \tag{4.10}$$

which is (4.9) without the differential part. Equation (4.10) can be thought of as an implicit expression for the variable z in terms of the variable θ, which one can rewrite as an explicit expression $z = h(\theta) \in \Bbbk((\theta^{1/s}))$ for the variable z. Note that $h(\theta)$ is not the same as the operator z. Since the leading term of f is $az^{-s/r}$, (4.10) implies that

$h(\theta) = a^{r/s}(-\theta)^{r/s} + \underline{o}(\theta^{r/s})$. Similar reasoning and (4.9) indicate that the operator z will be of the form

$$z = h(\theta) + *(-\theta)^{(r+s)/s} + \underline{o}(\theta^{(r+s)/s}). \tag{4.11}$$

Here the $* \in \Bbbk$ represents the coefficient that will arise from the interaction of the linear and differential parts of the operator θ. We wish to find the value for *. Let $A = f$ and $B = z\frac{d}{dz}$, then $[B, A] = zf'$. From (4.9) we have $-\theta = (A + B)^{-1}$, and we apply the Operator-root Lemma (1.5.3) to find

$$\begin{aligned}
(-\theta)^{\frac{r}{s}} &= f^{\frac{-r}{s}} - \left(\frac{r}{s}\right) f^{\frac{-r}{s}-1} z \left(\frac{\mathbb{Z}}{zr}\right) - \frac{1}{2}\left(\frac{r}{s}\right)\left(-\frac{r}{s} - 1\right) f^{\frac{-r}{s}-2} z f' + \underline{o}(z^{(r+s)/r}) \\
&= (a^{-r/s}z + \dots) + a^{-(r+s)/s}\left(\frac{-\mathbb{Z}}{s} + \frac{-(r+s)}{2s}\right) z^{(r+s)/r} + \underline{o}(z^{(r+s)/r}) \\
&= a^{-r/s}\Big(z + \cdots + a^{-1}\left[\frac{-\mathbb{Z}}{s} + \frac{-(r+s)}{2s}\right] z^{1+(s/r)} + \underline{o}(z^{1+(s/r)})\Big)
\end{aligned} \tag{4.12}$$

and

$$(-\theta)^{(r+s)/s} = a^{-1-(r/s)} z^{1+(s/r)} + \underline{o}(z^{1+(s/r)}). \tag{4.13}$$

Remark. We use the notation $\frac{\mathbb{Z}}{zr}$ to represent $\frac{d}{dz}$ since the operator $\frac{d}{dz} : K_r \to K_r$ acts on $z^{n/r}$ as $\frac{n}{rz}$ for all $n \in \mathbb{Z}$.

We can now find the value for * as follows. Substituting the expressions from (4.12) and (4.13) into (4.11) and making a short calculation indicates that * must be the value such that

$$* \, a^{-(r/s)-1} + a^{-1}\left[\frac{-\mathbb{Z}}{s} + \frac{-(r+s)}{2s}\right] = 0. \tag{4.14}$$

Thus

$$* = a^{r/s}\left[\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\right]$$

and we have the following expression for z:

$$z = h(\theta) + a^{r/s}\left[\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\right](-\theta)^{(r+s)/s} + \underline{o}(\theta^{(r+s)/s}). \tag{4.15}$$

According to (4.15), let us express $\hat{g}(\theta)$ as

$$\hat{g}(\theta) = h(\theta) - (-a)^{r/s}\left[\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\right]\theta^{(r+s)/s} + \underline{o}(\theta^{(r+s)/s}). \tag{4.16}$$

We now explain how to reduce $\hat{g}$ to the expression given for g in Theorem 4.1.1. According to Proposition 1.2.5, (1), $M_{\hat{g}}$ will be isomorphic to M_g where g is the expression given in (4.16) but without the $\underline{o}$ term. Moreover, one can add to g any expression of the form $(-a^{r/s})\frac{n}{s}\theta^{(r+s)/s}$ where $n \in \mathbb{Z}$. Thus we can eliminate the expression $\frac{\mathbb{Z}}{s}$ in (4.16) and write

$$g(\theta) = h(\theta) - (-a^{r/s})\left(\frac{r+s}{2s}\right)\theta^{(r+s)/s}. \tag{4.17}$$

Since $h(\theta) = z$, this matches equation (4.2) from Theorem 4.1.1. □

4.3.2. Proof of Theorem 4.1.3.

Proof. Given $\theta = -(z\nabla)^{-1}$ and $\nabla = \frac{d}{dz_x} + z_x^{-1}f$, we write $z = z_x + x$ and find that

$$\begin{aligned} -\theta &= \left[(x+z_x)\left(\frac{d}{dz_x} + z_x^{-1}f\right)\right]^{-1} \\ &= \left(zz_x^{-1}f + x\frac{d}{dz_x} + z_x\frac{d}{dz_x}\right)^{-1} \end{aligned} \tag{4.18}$$

Thus in the expression for $-\theta^{-1}$ there are three terms.

Case One: Regular singularity.

In this case we have $f = \alpha \in \Bbbk - \{0\}$, $s = 0$ and $r = 1$. As in Case One of the proof of Theorem 2.2.1, because α is only defined up to a shift by $\mathbb{Z}$ we can ignore the $\frac{d}{dz_x}$ term. The remaining portion of the proof is as described in the remark following the outline in subsection 4.3. Note that since $s = 0$, the extra θ term in (4.4) will vanish.

Case Two: Irregular singularity.

In this situation we have $\operatorname{ord}(f) < 0$. As we shall see in the proof, the only terms in (4.18) that affect the final result are those of order less than or equal to -1 (with respect to z_x). Specifically, since $z_x \frac{d}{dz_x}$ has order zero, all terms derived from it in the course of the calculations will fall into the $\underline{o}(\theta)$ term. Thus we can safely ignore the term $z_x \frac{d}{dz_x}$ for the remainder of the proof and consider only

$$-\theta = \left(z z_x^{-1} f + x \frac{d}{dz_x} \right)^{-1}. \tag{4.19}$$

We wish to express the operator z in terms of the operator θ. Our method will be similar to the proof of Theorem 4.1.1, but we first solve for $z_x = z - x$ in terms of θ, then add x to both sides to get an equation for z alone. Consider the equation

$$-\theta = (z z_x^{-1} f)^{-1}, \tag{4.20}$$

which is (4.19) without the differential part. Equation (4.20) can be thought of as an implicit expression for the variable z_x in terms of θ (note that we are thinking of z as $z_x + x$), which one can rewrite as an explicit expression $z_x = h(\theta) \in \Bbbk((\theta^{1/(r+s)}))$ for the variable z_x. This is the purely algebraic calculation which in the theorem is stated as explicitly expressing z_x in terms of $\theta^{1/(r+s)}$. Note that $h(\theta)$ represents the variable z_x (as defined by (4.20)), which is not the same as the operator z_x. Since the leading term of $z z_x^{-1} f$ is $x a z_x^{-(r+s)/r}$, (4.20) implies that $h(\theta) = (-ax)^{r/(s+r)} \theta^{r/(s+r)} + \underline{o}(\theta^{r/(s+r)})$. Using (4.19) we find that the operator z will be of the form

$$z_x = h(\theta) + *\theta + \underline{o}(\theta). \tag{4.21}$$

Here the * represents the coefficient that will arise from the interaction of the linear and differential parts of θ. We wish to find the value for *. For ease of computation, we multiply both sides of (4.19) by x and let $A = \frac{1}{x} z z_x^{-1} f$ and $B = \frac{d}{dz_x}$, so $[B, A] = A'$. Note that the leading term of A is $a z_x^{-(r+s)/r}$. From (4.19) we have $-x\theta = (A + B)^{-1}$, and we apply the Operator-root Lemma (1.5.3) to find

(4.22)

$$\begin{aligned}(-x\theta)^{\frac{r}{s+r}} &= A^{\frac{-r}{s+r}} - \frac{r}{s+r}A^{\frac{-r}{s+r}-1}\left(\frac{\mathbb{Z}}{z_x r}\right) + \frac{-r}{2(s+r)}\left(\frac{-r}{s+r}-1\right)A^{\frac{-r}{s+r}-2}A' + \dots \\ &= (a^{-r/(s+r)}z_x + \dots) + a^{-r/(s+r)-1}\left(\frac{-\mathbb{Z}}{s+r} + \frac{-2r-s}{2(s+r)}\right)z_x^{1+(s/r)} + \underline{o}(z_x^{1+(s/r)}) \\ &= a^{(-r/s+r)}\left(z_x + \dots + a^{-1}\left[\frac{-\mathbb{Z}}{s+r} + \frac{-2(r+s)+s}{2(s+r)}\right]z_x^{1+(s/r)} + \underline{o}(z_x^{1+(s/r)})\right) \\ &= a^{(-r/s+r)}\left(z_x + \dots + a^{-1}\left[\frac{-\mathbb{Z}}{s+r} - 1 + \frac{s}{2(s+r)}\right]z_x^{1+(s/r)} + \underline{o}(z_x^{1+(s/r)})\right)\end{aligned}$$

which implies that

$$(4.23)\qquad (-ax\theta)^{r/(s+r)} = z_x + \dots + a^{-1}\left(\frac{-\mathbb{Z}}{s+r} - 1 + \frac{s}{2(s+r)}\right)z^{1+(s/r)} + \underline{o}(z^{1+(s/r)}).$$

We also have

$$(4.24)\qquad \theta = \frac{-1}{ax}z_x^{1+(s/r)} + \underline{o}(z^{1+(s/r)}).$$

Remark. We use the notation $\frac{\mathbb{Z}}{z_x r}$ to represent $\frac{d}{dz_x}$ since the operator $\frac{d}{dz_x} : K_r \to K_r$ acts on $z_x^{n/r}$ as $\frac{n}{rz_x}$ for all $n \in \mathbb{Z}$.

We can now find the value for * as follows. Substituting the expressions from (4.24) and (4.23) into (4.21) and making a short calculation indicates that * must be the value such that

$$(4.25)\qquad *\left(\frac{-1}{ax}\right) + a^{-1}\left[\frac{-\mathbb{Z}}{s+r} - 1 + \frac{s}{2(s+r)}\right] = 0.$$

Thus

$$* = \frac{x\mathbb{Z}}{s+r} + \frac{xs}{2(s+r)}$$

and we have the following expression for z_x:

$$(4.26)\qquad z_x = h(\theta) + \left(\frac{x\mathbb{Z}}{s+r} + \frac{xs}{2(s+r)}\right)\theta + \underline{o}(\theta).$$

Letting $z_x = z - x$ gives

$$z = x + h(\theta) + \left(\frac{x\mathbb{Z}}{s+r} + \frac{xs}{2(s+r)}\right)\theta + \underline{o}(\theta). \tag{4.27}$$

According to (4.27), we express $\hat{g}(\theta)$ as

$$\hat{g}(\theta) = x + h(\theta) + \left(\frac{x\mathbb{Z}}{s+r} + \frac{xs}{2(s+r)}\right)\theta + \underline{o}(\theta). \tag{4.28}$$

We wish to reduce $\hat{g}$ to the expression given for g in (4.4). According to Proposition 1.2.5, (1) $D_{\hat{g}}$ will be isomorphic to D_g where g is the expression given in (4.28) but without the $\underline{o}$ term. Moreover, one can add to g any expression of the form $x\left(\frac{n}{s+r}\right)\theta$ where $n \in \mathbb{Z}$. Thus we can eliminate the expression $\frac{x\mathbb{Z}}{s+r}$ in (4.28) and write

$$g(\theta) = x + h(\theta) + \frac{xs}{2(s+r)}\theta. \tag{4.29}$$

Since $h(\theta) = z_x$ and $x + z_x = z$, this matches equation (4.4) from Theorem 4.1.3.

$\square$

4.3.3. Proof of Theorem 4.1.4.

Proof. Recall that $z = \frac{1}{\zeta}$ and $f \in \Bbbk((\zeta))$. Given $\theta = -(z\nabla)^{-1}$ and $\nabla = -\zeta^2\frac{d}{d\zeta} + \zeta f$, we find that

$$-\theta = \left(-\zeta\frac{d}{d\zeta} + f\right)^{-1}. \tag{4.30}$$

We wish to express the operator z in terms of the operator θ. First we find an expression for ζ in terms of θ, and then we will invert it.

Consider the equation

$$-\theta = f^{-1} \tag{4.31}$$

which is (4.30) without the differential part. Equation (4.31) can be thought of as an implicit expression for the variable ζ in terms of θ, which one can rewrite as an explicit expression $\zeta = h(\theta) \in \Bbbk((\theta^{1/s}))$ for the variable ζ. Since the leading term of f is $a\zeta^{-s/r}$, (4.31) implies that $h(\theta) = a^{r/s}(-\theta)^{r/s} + \underline{o}(\theta^{r/s})$. Using (4.30) we find that the operator ζ will be of the form

$$\zeta = h(\theta) + *\theta^{(r+s)/s} + \underline{o}(\theta^{(r+s)/s}). \tag{4.32}$$

Here the $* \in \Bbbk$ represents the coefficient that will arise from the interaction of the linear and differential parts of the operator θ. We wish to find the value for $*$. Let $A = f$ and $B = -\zeta\frac{d}{d\zeta}$, then $[B, A] = -\zeta f'$. From (4.30) we have $-\theta = (A + B)^{-1}$, and we apply the Operator-root Lemma (1.5.3) to find

(4.33)

$$\begin{aligned}
(-\theta)^{r/s} &= (A+B)^{-r/s} \\
&= A^{-r/s} - \frac{r}{s}A^{-(r/s)-1}B + \frac{1}{2}\left(\frac{-r}{s}\right)\left(\frac{-r}{s} - 1\right)A^{-(r/s)-2}[B, A] + \underline{o}(\zeta^{1+(r/s)}) \\
&= f^{-r/s} - \frac{r}{s}f^{(-r/s)-1}\left(-\zeta\frac{\mathbb{Z}}{\zeta r}\right) - \frac{r}{s}\left(\frac{1}{2}\right)\left(-\frac{r}{s} - 1\right)f^{-(r/s)-2}(-\zeta)f' + \underline{o}(\zeta^{1+(r/s)}) \\
&= (a^{-r/s}\zeta + \dots) + a^{-(r+s)/s}\left(\frac{-\mathbb{Z}}{s} + \frac{r+s}{2s}\right)\zeta^{(r+s)/s} + \underline{o}(\zeta^{1+(r/s)}) \\
&= a^{-r/s}\Big(\zeta + \cdots + a^{-1}\left[\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\right]\zeta^{1+(r/s)} + \underline{o}(\zeta^{1+(r/s)})\Big)
\end{aligned}$$

and

$$(-\theta)^{(r+s)/s} = a^{-1-(r/s)}\zeta^{1+(r/s)} + \underline{o}(\zeta^{1+(r/s)}). \tag{4.34}$$

Remark. We use the notation $\frac{\mathbb{Z}}{\zeta r}$ to represent $\frac{d}{d\zeta}$ since the operator $\frac{d}{d\zeta} : K_r \to K_r$ acts on $\zeta^{n/r}$ as $\frac{n}{r\zeta}$ for all $n \in \mathbb{Z}$.

We can now find the value for $*$ as follows. Substituting the expressions from (4.33) and (4.34) into (4.32) and making a short calculation indicates that $*$ must be the value such that

$$*(-a)^{-(r/s)-1} + a^{-1}\left(\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\right) = 0.$$

Thus

$$* = (-a)^{r/s}\left(\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\right)$$

and we have the following expression for ζ:

$$\zeta = h(\theta) + (-a^{r/s})\left(\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\right)\theta^{1+(r/s)} + \underline{o}(\theta^{1+(r/s)}). \tag{4.35}$$

We now invert both sides of (4.35) to get an expression for z in terms of θ. This gives

$$z = h(\theta)^{-1} - (-a)^{-r/s}\left(\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\right)\theta^{1-(r/s)} + \underline{o}(\theta^{1-(r/s)}). \tag{4.36}$$

According to (4.36), let us express $\hat{g}(\theta)$ as

$$\hat{g}(\theta) = h(\theta)^{-1} - (-a)^{-r/s}\left(\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\right)\theta^{1-(r/s)} + \underline{o}(\theta^{1-(r/s)})$$

and note that the leading term of $h(\theta)^{-1}$ is $(-a)^{-r/s}\theta^{-r/s}$. As in the proof for Theorem 4.1.1, we can choose an appropriate isomorphic connection to give us

$$g(\theta) = h(\theta)^{-1} - (-a)^{-r/s}\left(\frac{r+s}{2s}\right)\theta^{1-(r/s)}.$$

Since $h(\theta)^{-1} = z$, this matches the expression found in Theorem 4.1.4.

$\square$

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